Available online at www.sciencedirect.com
ScienceDirect
Journal of Computational and Applied Mathematics 223 (2009) 421–437
JOURNAL OF COMPUTATIONAL AND APPLIED MATHEMATICS
www.elsevier.com/locate/cam

The analysis of a finite element method for the three-species Lotka–Volterra competition-diffusion with Dirichlet boundary conditions

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Received 13 August 2007; received in revised form 28 December 2007

Abstract

A Galerkin finite element method is developed for the two dimensional/three dimensional nonlinear time-dependent three-species Lotka–Volterra competition-diffusion equations on a bounded domain. The existence and uniqueness of the solution to the numerical formulation are proved. An error estimate for the numerical solution is obtained. Numerical computations are carried out to examine the expected orders of accuracy in the error estimates.

Keywords: Three-species Lotka–Volterra competition-diffusion equations; Finite element; Error estimate; Two-dimensional; Three dimensional

1. Introduction

In this paper, we assume that all competing species occur by diffusion. Let Ω be a bounded domain in Euclidean space $\mathbb{R}^d$ ($d = 2$ or 3) with a piecewise smooth boundary $\partial\Omega$. We restrict ourselves to considering a square/cubic domain Ω under the Dirichlet boundary conditions. A fixed final time is denoted by T. The model treated here is of the form

$$\begin{cases} \dfrac{\partial A_1}{\partial t} = d_1\nabla^2 A_1 + A_1(r_1 - a_{11}A_1 - a_{12}A_2 - a_{13}A_3), & \text{in } \Omega \times (0, T], \\ \dfrac{\partial A_2}{\partial t} = d_2\nabla^2 A_2 + A_2(r_2 - a_{21}A_2 - a_{22}A_3 - a_{23}A_1), & \text{in } \Omega \times (0, T], \\ \dfrac{\partial A_3}{\partial t} = d_3\nabla^2 A_3 + A_3(r_3 - a_{31}A_3 - a_{32}A_1 - a_{33}A_2), & \text{in } \Omega \times (0, T], \end{cases} \tag{1}$$

where

- t is the time(s) and $\boldsymbol{x} = (x, y)$ or $\boldsymbol{x} = (x, y, z)$ is a function of 2D position or 3D position in the Cartesian coordinate system, respectively;

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0377-0427/$ - see front matter
doi:10.1016/j.cam.2008.01.030

- $A_i(\boldsymbol{x}, t)$, $1 \le i \le 3$, present the population density of the ith species at $\boldsymbol{x}$ and t;
- d_i, $1 \le i \le 3$, are the positive diffusion coefficients;
- r_i, $1 \le i \le 3$, are the positive intrinsic growth rates of ith population;
- a_{ij}, $1 \le i, j \le 3$, are the positive coefficients accounting for the intra-specific competition if $i = j$, and the inter-specific competition if $i \ne j$.

Dirichlet boundary conditions on the three competing species are:

$$A_i(\boldsymbol{x}, t) = 0, \quad 1 \le i \le 3, \text{ on } \partial\Omega \times [0, T]. \tag{2}$$

Initial conditions are:

$$A_i(\boldsymbol{x}, t = 0) = A_{i,0}(\boldsymbol{x}), \quad 1 \le i \le 3, \boldsymbol{x} \in \bar{\Omega}, \tag{3}$$

where $A_{i,0}$, $1 \le i \le 3$, are the three prescribed competing species values; $\bar{\Omega} = \Omega \cup \partial\Omega$ will denote the closure of Ω.

The diffusive 3-species Lotka–Volterra (LV) systems as discussed by several investigators, e.g., [1–3] have been an active field of inquiry that mimic the population dynamics of interacting species because of their applications in the predator–prey system and mathematical biological models. The positive coefficients a_{ij}, $1 \le i, j \le 3$ play an important role in characterizing the 3-species competitive systems. Kishimoto [4] studied a stable non-constant equilibrium solution of the diffusive Lotka–Volterra system with three species. Kan-on [5] studied the existence and instability of Neumann layer solutions for a 3-component Lotka–Volterra model with diffusion. The effect of diffusion for the multispecies Lotka–Volterra competition model was investigated by Martínez [6]. For the cross-diffusion case, see [7].

It is worth mentioning that the diffusive 3-species Lotka–Volterra systems presented here have a tie to the classical Gause–Lotka–Volterra equations that consist of a set of time-dependent ordinary differential equations with nonlinear quadratic terms, e.g. [8,9]. Because of the sign of a_{ij}, $1 \le i, j \le 3$, Frachebourg et al. [10] investigated the spatial organization in cyclic Lotka–Volterra systems. Gyllenberg et al. [11,12] also studied limit cycles for competitor–competitor–mutualist Lotka–Volterra systems.

In this paper, we construct the semi-implicit finite element (FE) scheme for (1)–(3) in Section 2. The solvability of the proposed scheme will be demonstrated in Section 3. We make use of theory from prior estimates and techniques. Optimal order estimates in the H^1 norm are derived for the errors in the approximate solution. Error estimates will be given in Section 4. A second-order semi-implicit scheme in time for the LV equations will be given in Section 5. In Section 6, a series of numerical computations are carried out to examine the expected orders of accuracy in the error estimates. A few remarks will be given in Section 7.

2. Weak formulation and finite element formulation

2.1. Weak formulation

To obtain the weak formulation for the problem (1)–(3), we introduce some notation. We denote by $H^k(\Omega)$ the usual Sobolev space containing functions having the finite norm

$$\|u\|_k = \left[\int_\Omega \sum_{\alpha \le k} \left| D^\alpha u \right|^2 \mathrm{d}\boldsymbol{x} \right]^{1/2}.$$

Similarly,

$$\|u\|_\infty = \sup_{\boldsymbol{x} \in \Omega} |u(\boldsymbol{x})|.$$

Let

$$H_0^1(\Omega) = \{u | u = 0 \text{ on } \partial\Omega\}.$$