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A smoothing Newton method for a type of inverse semi-definite quadratic programming problem

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Abstract

We consider an inverse problem arising from the semi-definite quadratic programming (SDQP) problem. We represent this problem as a cone-constrained minimization problem and its dual (denoted ISDQD) is a semismoothly differentiable (SC^1) convex programming problem with fewer variables than the original one. The Karush–Kuhn–Tucker conditions of the dual problem (ISDQD) can be formulated as a system of semismooth equations which involves the projection onto the cone of positive semi-definite matrices. A smoothing Newton method is given for getting a Karush–Kuhn–Tucker point of ISDQD. The proposed method needs to compute the directional derivative of the smoothing projector at the corresponding point and to solve one linear system per iteration. The quadratic convergence of the smoothing Newton method is proved under a suitable condition. Numerical experiments are reported to show that the smoothing Newton method is very effective for solving this type of inverse quadratic programming problems.

MSC: 90C20; 90C22

Keywords: Semi-definite quadratic programming; Inverse optimization; Smoothing Newton method

1. Introduction

A typical optimization problem is a forward problem, in which there are usually parameters associated with decision variables in the objective function and constraints. When solving the typical optimization problem, the values of these parameters usually are available and we need to find an optimal solution to it. An inverse optimization problem is to find values of parameters which make the known solutions optimal and which differ from the given estimates as little as possible.

The interest in inverse optimization problems was initiated by the paper [5] dealing with an inverse shortest path problem. In the past few years, a variety of inverse combinatorial optimization problems have been studied by researchers, see, for example, the survey paper [8] and the references [1,2,4,6,20], etc. But for continuous optimization,

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there are just a few papers on their inverse problems, except for linear programming [18,19] and for quadratic programming [21].

In this paper, we consider a semi-definite quadratic programming problem of the form

$$\begin{aligned}&\text{SDQP}(G, c, \mathcal{A}, B)\\ &\min \quad f(x) := \frac{1}{2}x^T Gx + c^T x \\ &\text{s.t.} \quad x \in \Omega_P := \{x' \in \mathbf{R}^n \mid \mathcal{A}x' \preceq B\},\end{aligned} \tag{1.1}$$

where $G \in \mathcal{S}^n_+$, $\mathcal{S}^n$ denotes the space of $n \times n$ symmetric matrices, $\mathcal{S}^n_+$ denotes the cone of $n \times n$ positive semi-definite symmetric matrices. For any $C, D \in \mathcal{S}^n$, denote $\text{Tr}(C)$ the trace of C, $\langle C, D\rangle = \text{Tr}(C^T D)$, $\|C\|_F = \sqrt{\langle C, C\rangle}$, $C \succeq D$ if and only if $C - D \in \mathcal{S}^n_+$. $\mathcal{A} : \mathbf{R}^n \to \mathcal{S}^m$ is a linear operator and $\mathcal{A}^* : \mathcal{S}^m \to \mathbf{R}^n$ is the adjoint of $\mathcal{A}$, $c \in \mathbf{R}^n$ and $B \in \mathcal{S}^m$. we define $\mathcal{A}$ by

$$\mathcal{A}x := \sum_{j=1}^{n} x_j A_j, \quad \forall x \in \mathbf{R}^n,$$

then $\mathcal{A}^*$ is defined by

$$\mathcal{A}^*(X) := \begin{bmatrix} \langle A_1, X\rangle \\ \langle A_2, X\rangle \\ \vdots \\ \langle A_n, X\rangle \end{bmatrix}, \quad \forall X \in \mathcal{S}^m,$$

where $A_i \in \mathcal{S}^m$ for $i = 1, \ldots, n$. For simplicity of notations, we introduce "SOL" as a mapping whose variables are problems, we denote SOL(P) to be the set of optimal solutions to a problem (P).

Given a feasible point $x^0 \in \Omega_P$, which should be the optimal solution to Problem $\text{SDQP}(G, c, \mathcal{A}, B)$ and a pair $(G^0, c^0) \in \mathcal{S}^n \times \mathbf{R}^n$ which is an estimate to (G, c). The inverse semi-definite quadratic programming (ISDQP) considered in this paper is to find a pair $(G, c) \in \mathcal{S}^n_+ \times \mathbf{R}^n$ to solve

$$\begin{aligned}&\text{ISDQP}(\mathcal{A}, B)\\ &\min \quad \frac{1}{2}\|(G, c) - (G^0, c^0)\|^2 \\ &\text{s.t.} \quad x^0 \in \text{SOL}(\text{SDQP}(G, c, \mathcal{A}, B)), \\ &\qquad (G, c) \in \mathcal{S}^n_+ \times \mathbf{R}^n,\end{aligned} \tag{1.2}$$

where $\|\cdot\|$ is defined by $\|(G', c')\| := \sqrt{\text{Tr}(G'^T G') + c'^T c'}$ for $(G', c') \in \mathcal{S}^n \times \mathbf{R}^n$.

Problem (1.2) is a cone-constrained optimization problem with a quadratic objective function. The scale of this problem will be quite large when n is a large number as the number of its decision variables is $n + n(n + 1)/2$. Our main idea in this paper is that, instead of dealing with Problem (1.2) directly, we focus on solving its dual problem. The reason for doing this is that the dual is a SC^1 convex programming problem with fewer ($\leq n$) decision variables than the original inverse quadratic problem, and its feasible set is a SDP cone. We consider the smoothing Newton method, developed by [17], for getting a Karush–Kuhn–Tucker point of the dual problem.

Throughout this paper the following notations will be used. We denote the symmetric square root of X by $X^{\frac{1}{2}}$. Let $|X| := (X^2)^{\frac{1}{2}}$ and $\Pi_{\mathcal{S}^n_+}(X) := (X + |X|)/2$ for any $X \in \mathcal{S}^n$. The Hadamard product of X and Y is denoted by $X \circ Y$, namely $(X \circ Y)_{ij} := X_{ij} Y_{ij}$. Let I be the identity matrix of appropriate dimension.

This paper is organized as follows. In Section 2, we give some results from nonsmooth analysis which shall be used in our convergence analysis. Section 3 is devoted to deriving the dual of the inverse quadratic programming problem. In Section 4, we describe the smoothing Newton method for problem (3.9) and prove the global convergence and the quadratic convergence rate. Numerical results implemented by the smoothing Newton method are given in Section 5.