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Rao's statistic for constant and proportional hazard models ☆

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ABSTRACT

In this paper we introduce a new family of measures of divergence for the analysis of the degree of departure from a model with a constant hazard function and also for comparing if two models have proportional hazard rates. Our family of measures is based on the family of divergences introduced by Burbea and Rao (see [J. Burbea, C.R. Rao, On the convexity of higher order Jensen differences based on entropy functions, IEEE Transactions on Information Theory 28 (1982) 961–963]). Some well-known sets of data are reanalyzed using the new families of test statistics and confidence intervals introduced in this paper.

1. Introduction

Let X be a nonnegative random variable representing the lifetime of individuals in some population. Usually X is assumed to be continuous. Sometimes, for example, when lifetimes are grouped or when 'lifetime' refers to an integer number of cycles of some sort, it may be desired to treat X as a discrete random variable. Suppose X can take the values $x_1, \ldots, x_k$ with $0 \leq x_1 \leq x_2 \leq \cdots \leq x_k$ and let the probability function

$$p_j = \Pr(X = x_j), \quad j = 1, 2, \ldots, k,$$

that is, p_j represents the probability that an individual fails at time x_j and we denote $\boldsymbol{p} = (p_1, \ldots, p_k)^T$. The survival function is then

$$S(x) = \Pr(X \geq x) = \sum_{\{j: x_j \geq x\}} p_j$$

and the Hazard function at x_i is now defined by

$$\theta_i = \Pr(X = x_i / X \geq x_i) = p_i \Big/ \sum_{j=i}^{k} p_j, \quad i = 1, \ldots, k-1.$$

Models with a constant hazard function are important and have a particularly simple structure, i.e., $\theta_i = \lambda, \ i = 1, \ldots, k-1$. In relation to the Hazard function it is possible to consider the following probability vector

$$\boldsymbol{v} = (v_1, \ldots, v_{k-1})^T = \left(\frac{\theta_1}{\tau}, \ldots, \frac{\theta_{k-1}}{\tau}\right)^T,$$

where $\tau = \sum_{i=1}^{k-1} \theta_i$ which for a model with a constant Hazard function is given by

$$\boldsymbol{v}_0 = \left(v_1^0, \ldots, v_{k-1}^0\right)^T = (1/(k-1), \ldots, 1/(k-1))^T.$$

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Bhattacharya [1] proposed a measure to represent the degree of departure from a model with a constant hazard function based on the information contained in the vector $\boldsymbol{v}$ and using the Kullback-divergence. In the same paper he also introduced a measure to represent the degree of departure from proportional hazard rates when two systems are present and grouped data are considered, also based on the Kullback-divergence. Later Bhattacharya [2] proposed another measure to represent the degree of departure from a model with a constant hazard function based on information contained in the vector $\boldsymbol{v}$ and on the basis of the well-known ϕ-divergence measures. He presented a test statistic for testing if a model has a constant Hazard function; the asymptotic distribution of this test statistic is a linear combination of chi-square random variables. This fact is interesting because in most of the statistical problems test statistics based on Kullback-leibler divergence in particular, and on ϕ-divergence measures in general, have an asymptotic chi-square distribution. Therefore in some problems, for instance in goodness-of -fit there is an advantage in considering test statistics based on ϕ-divergences instead other family of divergence measures, i.e., R_ϕ-divergences in [19]. In the problem considered in this paper this advantage disappears because the family of test statistics considered in [1,2] does not have an asymptotic chi-square distribution. This is the main motivation in considering R_ϕ-divergences in this paper in order to get test statistics for testing if the hazard function is constant or the models have proportional hazard rates.

The family of R_ϕ-divergences was introduced in [19] and studied later in [4,5,3]. This family of divergence measures is defined for two different models characterized by the probability distributions $\boldsymbol{v}_1 = (v_{11}, \ldots, v_{1,k-1})^T$ and $\boldsymbol{v}_2 = (v_{21}, \ldots, v_{2,k-1})^T$ with $v_{ij} = \theta_{ij}/\tau_i$, $\tau_i = \sum_{l=1}^{k-1} \theta_{il}$ and θ_{ij} the Hazard function at $x_j, j = 1, \ldots, k-1$, for the model i, $i = 1, 2$, by

$$R_\phi(\boldsymbol{v}_1, \boldsymbol{v}_2) = H_\phi\left(\frac{\boldsymbol{v}_1 + \boldsymbol{v}_2}{2}\right) - \frac{1}{2}\left(H_\phi(\boldsymbol{v}_1) + H_\phi(\boldsymbol{v}_2)\right), \tag{1}$$

where

$$H_\phi(\boldsymbol{v}_i) = \sum_{j=1}^{k-1} \phi\left(v_{ij}\right), \quad i = 1, 2,$$

is the ϕ-entropy associated with the probability distribution $\boldsymbol{v}_i = (v_{i1}, \ldots, v_{i,k-1})^T, i = 1, 2$ and $\phi : (0, \infty) \to \mathbb{R}$ a continuous concave function with $\phi(0) = \lim_{t\to 0} \phi(t) \in (-\infty, \infty)$. Some interesting properties of ϕ-entropies can be seen in [23,22,16]. The convexity of the R_ϕ-divergences holds if the function $\phi(x)$ is concave and $\phi''(x)^{-1}$ is convex. Some properties of this family of divergences can be seen in the cited papers of Burbea and Rao [4,5] and Pardo and Vajda [18]. Important applications of the R_ϕ-divergences in estimation and testing can be found in [17,14].

An important family of R_ϕ-divergences is obtained if we consider the entropies of degree a due to [8]

$$\phi_a = \begin{cases} (1-a)^{-1}(x^a - x), & a \neq 1 \\ -x \log x & a = 1. \end{cases} \tag{2}$$

Rao [19] used the family of ϕ_a-entropies in genetic diversity between populations. For $a = 2$, the associated R_ϕ-divergence is proportional to the square of the euclidean distance

$$R_{\phi_2}(\boldsymbol{v}_1, \boldsymbol{v}_2) = \frac{1}{4}\sum_{i=1}^{k-1}(v_{1i} - v_{2i})^2.$$

Another important family of R_ϕ-divergences is obtained if we consider the Bose–Einstein entropy introduced in [4] or Fermi–Dirac entropy, [10].

Based on $R_\phi(\boldsymbol{v}_1, \boldsymbol{v}_2)$ defined in (1), in this paper we introduce a class of measures to represent the degree of departure from a model with a constant hazard function, as well as, another class of measures to represent the degree of departure from the proportional hazards rate model when two systems are present. In Section 2 we study the measure of departure from the constant hazard model and the asymptotic distribution of the new family of test statistics based on R_ϕ-divergence when the model has no constant Hazard function and when the model has constant Hazard function. Based on these asymptotic results we present some families of test statistics for testing if the model has constant Hazard function and some confidence intervals. In Section 3 we study the measure of departure from proportional hazard models when two systems are present based on (1) and we find the asymptotic distribution of the corresponding statistics. Finally in Section 4 we present some conclusions about the results obtained in Sections 2 and 3.

2. Measure of departure from constant hazard model

The degree of departure from a model with a constant hazard function is defined on the basis of the R_ϕ-divergence measures by

$$R_\phi(\boldsymbol{v}, \boldsymbol{v}_0) = \sum_{l=1}^{k-1} \phi\left(\frac{v_l + (1/(k-1))}{2}\right) - \frac{1}{2}\left[\sum_{l=1}^{k-1}\phi(v_l) + \sum_{l=1}^{k-1}\phi\left(\frac{1}{k-1}\right)\right].$$

As usual we denote $\hat{\boldsymbol{p}} = (\hat{p}_1, \ldots, \hat{p}_k)^T$ with $\hat{p}_j = n_j/n$, where n_j denotes the observed frequency for the ith point and we assume that $(n_1, \ldots, n_k)$ is a realization of a multinomial sampling with parameters $(n, p_1, \ldots, p_k)$ where $n = \sum_{j=1}^k n_j$ is the number of individuals of which we are analyzing the lifetime. We denote by $R_\phi(\hat{\boldsymbol{v}}, \boldsymbol{v}_0)$ the new family of statistics based on R_ϕ-divergence obtained replacing $(p_1, \ldots, p_k)$ by $(\hat{p}_1, \ldots, \hat{p}_k)$ in $\theta_j, j = 1, \ldots, k-1$.