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Fractional evolution Dirac-like equations: Some properties and a discrete Von Neumann-type analysis

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ABSTRACT

A system of fractional evolution equations results from employing the tool of the Fractional Calculus and following the method used by Dirac to obtain his well-known equation from Klein–Gordon's one. It represents a possible interpolation between Dirac and diffusion and wave equations in one space dimension.

In this paper some analytical properties typical of the general solution of this system of equations are obtained and necessary stability bounds for a numerical scheme approximating such equations are found, through the classical discrete Von Neumann-type analysis.

The non-local property of the time fractional differential operator leads to discretizations in terms of series. Here, the analytical methods, usually employed in the study of the stability of discrete schemes when dealing with integer order differential equations, have been adapted to the complexity of the real order case.

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1. Introduction

In this paper, we will consider a generalization of the linear one-dimensional diffusion and wave equations, that we call "fractional evolution Dirac-like equations". They are obtained by combining the fractional derivatives and the internal degrees of freedom associated to a system, as we will explain below.

The Fractional Calculus (see [4,14,15], for example) deals with the theory of real (or imaginary) order integral and differential operators and it represents a natural instrument to model non-local phenomena, either in space or time, involving different scales.

The general fractional derivative in the variable x can be denoted by D_x^α, where $\alpha > 0$, $\alpha \in \mathbb{R}$, and it must coincide with the classical derivatives for integer orders α. Many different definitions have been proposed in the literature, all preserving this property. Here, we will refer to two of them, especially employed in the mathematical and physical fields.

The first is the Riemann–Liouville fractional derivative of order $\alpha > 0, \alpha \in \mathbb{R}$ (see [15], for example) of a function f given in $[a, b]$, where $[a, b] \subset \mathrm{R}$, $n \in \mathbb{N}$, $n = -[-\alpha]$ and $x > a$:

$$({}_a^{RL}D_x^\alpha f)(x) = \frac{\mathrm{d}^n}{\mathrm{d}x^n}\frac{1}{\Gamma(n-\alpha)}\int_a^x \frac{f(\tau)}{(x-\tau)^{\alpha-n+1}}\mathrm{d}\tau. \tag{1}$$

The second one is the Caputo fractional derivative, which can be considered as a regularised version of the previous definition, since it takes the form

$$({}_a^{C}D_x^\alpha f)(x) = \frac{1}{\Gamma(n-\alpha)}\int_a^x \frac{f^n(\tau)}{(x-\tau)^{\alpha-n+1}}\mathrm{d}\tau. \tag{2}$$

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The following relation between the above definitions holds:

$$({}_a^C D_x^\alpha f)(x) = ({}^{RL}D_{a+}^\alpha f)(x) - \sum_{j=0}^{n-1} \frac{f^{(j)}(a+)}{\Gamma(1+j-\alpha)}(x-a)^{j-\alpha}, \tag{3}$$

and a sufficient condition under which both derivatives exist is $f \in AC^{n-1}(a,b)$ and $f^n(x) \in L^1(a,b)$. Property (3) allows to establish the equivalence between null fractional and classical initial conditions. In fact, if (3) is valid, then:

$$({}^{RL}D_{a+}^\alpha f)(a+) = 0 \iff f^{(j)}(a+) = 0, \quad j = 0, 1, 2, \ldots, n-1. \tag{4}$$

Among the most important differences between the Riemann–Liouville and the Caputo type derivative is the fact that the first one is not zero when calculated on a constant function, whereas the second one is. This analogy with the classical derivative represents one of the main motivations which led Caputo in 1967 [1] to introduce in his studies about applied problems the operator (2) as an alternative to the Riemann–Liouville operator.

Another important distinction between the Riemann-Lioville and the Caputo operator is their behavior under the Laplace transform. In fact, assuming certain restrictions on the function $f(t) : \mathbb{R}_+ \to \mathbb{C}$ such that the following expressions exist, it results:

$$\int_0^\infty \mathrm{e}^{-st} \left(({}^{RL}D_0^\alpha f)(t)\right) \mathrm{d}t = s^\alpha (\mathcal{L}_t f)(s) - \sum_{k=0}^{n-1} s^k ({}^{RL}D_0^{\alpha-k-1} f)(0), \tag{5}$$

$$\int_0^\infty \mathrm{e}^{-st} \left(({}^{C}D_0^\alpha f)(t)\right) \mathrm{d}t = s^\alpha (\mathcal{L}_t f)(s) - \sum_{k=0}^{n-1} s^{\alpha-k-1} f^{(k)}(0), \tag{6}$$

where $s \in \mathbb{C}$ and $\mathcal{L}_t f$ is the Laplace transform of the function f.

Property (6) allows us to employ initial conditions of the classical type with the usual interpretation when solving applied problems through the Laplace transform, which is one of the most recurring method in the physical field due to its easy of use.

Let us now define the fractional evolution Dirac-like equations. It is important to highlight that in the construction of such a generalization of the linear one dimensional diffusion and wave equations, the definition of the fractional operator does not need to be specified from the very beginning. In fact, we will just have to know the fractional operator we are referring to when we want to calculate the solutions of the corresponding specific fractional evolution equations.

As it is known, the free Dirac equation is, in some sense, the square root of the Klein–Gordon equation (see, e.g., [17]). Similarly, we can consider a kind of square root of the following fractional diffusion equation in one space dimension:

$$(D_t^{2\alpha} u)(t,x) - \lambda^2 \partial_{xx} u(t,x) = 0, \tag{7}$$

which has been widely studied in the literature (see [7–10,16], for example).

To do that, let us consider the general system of fractional evolution Dirac-like equations

$$\left(\mathbf{A} D_t^\alpha + \lambda \mathbf{B} \partial_x\right) \mathbf{v}(t,x) = \mathbf{0}, \qquad \mathbf{v}(t,x) = \begin{pmatrix} u_1(t,x) \\ u_2(t,x) \end{pmatrix}, \tag{8}$$

with $0 < \alpha \leq 1$, $\alpha \in \mathbb{R}$, $\lambda \in \mathbb{R}$, $\lambda \neq 0$ and where $\mathbf{A}$ and $\mathbf{B}$ are 2×2 matrixes satisfying the Pauli's algebra:

$$\mathbf{A}^2 = \mathbf{I}, \qquad \mathbf{B}^2 = -\mathbf{I}, \qquad \mathbf{AB} + \mathbf{BA} = \mathbf{0} \tag{9}$$

and $\mathbf{I}$ is the identity matrix.

Each component of the solution $\mathbf{v}(t,x)$ also solves (7) provided the index property

$$({D_t}^\alpha {D_t}^\alpha)\mathbf{v}(t,x) = {D_t}^{2\alpha}\mathbf{v}(t,x) \tag{10}$$

holds.

Actually, under the assumption (10), it turns out that:

$$\begin{aligned} &\left(\mathbf{A} D_t^\alpha + \lambda \mathbf{B} \partial_x\right) \mathbf{v}(t,x) = \mathbf{0} \quad \Longrightarrow \\ &\left(\mathbf{A} D_t^\alpha + \lambda \mathbf{B} \partial_x\right)^2 \mathbf{v}(t,x) = \begin{cases} (D_t^{2\alpha} u_1)(t,x) - \lambda^2 \partial_{xx} u_1(t,x) = 0 \\ (D_t^{2\alpha} u_2)(t,x) - \lambda^2 \partial_{xx} u_2(t,x) = 0. \end{cases} \end{aligned} \tag{11}$$

Observe that, for the fractional Riemann–Liouville and Caputo derivatives the semigroup property (10) occurs when $\mathbf{v}(0,x) = \mathbf{0}$.

In fact, the following proposition holds (see, e.g., [14,15]):

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