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Numerical solution of the Orr–Sommerfeld equation using the viscous Green function and split-Gaussian quadrature

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ABSTRACT

We continue our study of the construction of numerical methods for solving two-point boundary value problems using Green functions, building on the successful use of split-Gauss-type quadrature schemes. Here we adapt the method for eigenvalue problems, in particular the Orr–Sommerfeld equation of hydrodynamic stability theory. Use of the Green function for the viscous part of the problem reduces the fourth-order ordinary differential equation to an integro-differential equation which we then discretize using the split-Gaussian quadrature and product integration approach of our earlier work along with pseudospectral differentiation matrices for the remaining differential operators. As the latter are only second-order the resulting discrete equations are much more stable than those obtained from the original differential equation. This permits us to obtain results for the standard test problem (plane Poiseuille flow at unit streamwise wavenumber and Reynolds number 10 000) that we believe are the most accurate to date.

1. Introduction

The solution of an inhomogeneous linear two-point boundary value problem generally has a Green integral representation [19, pp. 254–257; e.g.]; that is, if $\mathcal{L}$ is a linear ordinary differential operator then the solution ϕ of

$$\mathcal{L}\phi = f \tag{1}$$

satisfying appropriate boundary conditions at $x = a$ and $x = b$ can be written

$$\phi(x) = \int_a^b G(x,\xi) f(\xi)\,\mathrm{d}\xi, \quad (a \le x \le b). \tag{2}$$

If the Green function for a given operator and boundary conditions is known, this can be used as the basis of a stable and accurate method of computing the solution of the problem [26,27]. Here we extend this approach to generalized eigenvalue problems of the form

$$\mathcal{L}\phi + \mathcal{K}\phi = c\mathcal{M}\phi, \tag{3}$$

where c is the eigenvalue and $\mathcal{K}$ and $\mathcal{M}$ are linear ordinary differential operators of lower order than $\mathcal{L}$ and we assume that the Green function for the related two-point boundary value problem (1) is known. Formal application of (2) leads to

$$\phi(x) + \int_a^b G(x,\xi)(\mathcal{K}\phi)(\xi)\,\mathrm{d}\xi = c\int_a^b G(x,\xi)(\mathcal{M}\phi)(\xi)\,\mathrm{d}\xi. \tag{4}$$

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The hope is that the integro-differential equation (4) will lead to more stable discrete equations than the original ordinary differential equation (1), as numerical integration is usually more stable than numerical differentiation. In this paper we apply the approach to a famous eigenvalue problem of the form (3), the Orr–Sommerfeld equation of hydrodynamic stability [12, p. 156], and find that this is indeed the case.

1.1. The Orr–Sommerfeld equation

The flow of viscous incompressible fluids is usually described by the Navier–Stokes equations, which comprise a quasilinear second-order vector equation governing the momentum balance and a linear first-order scalar equation governing the conservation of mass [22, p. 2]. Although the system is difficult to solve in general configurations, one-dimensional domains such as the channel between two infinite parallel plates admit simple steady unidirectional solutions such as plane Poiseuille flow [22, p. 4]. Linearizing the momentum equation about such a solution, eliminating the pressure using the conservation of mass, and seeking solutions complex-exponential in time and the streamwise direction (noting that the system is homogeneous in these coordinates and that the equations have coefficients constant with respect to them) leads to the Orr–Sommerfeld equation [22, p. 7].

The Orr–Sommerfeld equation for a velocity profile $U(x)$, Reynolds number Re, streamwise wavenumber α, and wave speed c is [12, p. 156]

$$\phi^{\mathrm{iv}} - 2\alpha^2\phi'' + \alpha^4\phi = \mathrm{i}\alpha Re\left\{(U-c)\left(\phi'' - \alpha^2\phi\right) - U''\phi\right\}. \tag{5}$$

Here $\phi(x)$ is related to the stream-function perturbation $\delta\psi$ by $\delta\psi(x, y, t) = \phi(x)\exp\{\mathrm{i}\alpha(x-ct)\}$. The boundary conditions for a channel with solid walls at $x = \pm 1$ are

$$\phi(\pm 1) = 0 \quad \text{(‘impermeability’)} \tag{6a}$$

$$\phi'(\pm 1) = 0 \quad \text{(‘no-slip’)}. \tag{6b}$$

The Reynolds number Re is defined as the centre-line velocity times the channel width divided by the kinematic viscosity (in any consistent set of physical units); thus the left-hand side represents the effects of viscosity. Without viscosity, $Re^{-1} = 0$, the left-hand side is dropped, and only the part of (5) in braces remains: the Rayleigh equation [12, p. 130]. The Rayleigh equation is only second order, so the no-slip boundary conditions (6b) have to be dropped, and further the coefficient of the second derivative vanishes wherever $U(x) = c$; these singularities do not occur in the Orr–Sommerfeld equation, being smeared out by viscosity into boundary layers and critical layers, respectively. The Rayleigh and Orr–Sommerfeld equations have been the subject of much study in hydrodynamics [22,7,12].

Both the Orr–Sommerfeld and Rayleigh equations are homogeneous in ϕ, and are therefore usually solved as eigenvalue problems for the unknown complex c given real α (the ‘temporal’ problem) or complex α, given real frequency $-\alpha c$ (the ‘spatial’ problem) [12, p. 152]. Here we consider only the temporal problem, for which the left-hand side contains a fourth-order ordinary differential operator with known real constant coefficients.

For many velocity profiles of interest, such as plane Poiseuille flow for which $U(x) = 1 - x^2$, the physically relevant Reynolds numbers are quite large; for example, eigenvalues c with positive imaginary part, denoting exponential growth of disturbances in time, only occur for real wavenumbers α if $Re > 5772.2$ [12, p. 192]. This means that the boundary and critical layers are thin which implies difficulty in numerically solving (5). Many different approaches have been tried, including compact finite difference [33], Galerkin [11], shooting [30], Chebyshev-τ [31], stabilized shooting [8], and pseudospectral [18,34,25,24]. These diverse methods have achieved varying degrees of accuracy and efficiency.

To date, to our knowledge, the most accurate results have been obtained by the various ‘spectral’ methods [31,18]; however, we have found empirically that they are not numerically stable, that is, the error ultimately grows with the number of nodes or expansion terms. For many problems, this transition occurs at such a high number and the convergence is initially so rapid that it does not prevent very accurate results being obtained; nevertheless, numerically stable methods are more robust and reliable in practice, and so the search for one is not without value. In earlier studies [26,27] we found the Green function integral expression for the solution of a two-point boundary value problem to be very stable, and we developed quadrature techniques to render it accurate and rapidly convergent too. In this paper we adapt this approach to eigenvalue problems and apply it to the Orr–Sommerfeld equation for a standard test case: plane Poiseuille flow at $Re = 10^4$ and $\alpha = 1$ [33,14,31,36,4,18,34].

2. Recasting the Orr–Sommerfeld equation

2.1. The viscous Green function

As discussed by Li [21], the solution of the two-point boundary value problem

$$\phi^{\mathrm{iv}} - 2\alpha^2\phi'' + \alpha^4\phi = f, \tag{7}$$