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A fractional step θ-method approximation of time-dependent viscoelastic fluid flow

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ABSTRACT

A fractional step θ-method for the approximation of time-dependent viscoelastic fluid flow equations is described and analyzed in this article. The algorithm uses substeps within a time step to sequentially update velocity, pressure, and stress. This lagged approach to temporal integration requires a resolution of smaller systems than a fully implicit approach while achieving a second order temporal accuracy. We establish a priori error estimates for our scheme, and provide numerical computations to support the theoretical results and demonstrate the capability of this method.

1. Introduction

Modeling viscoelastic fluid flow is computationally difficult for a variety of reasons. Assuming *slow* flow, the modeling equations represent a "Stokes-like system" for the conservation of mass and momentum equations, coupled with a nonlinear hyperbolic equation describing the constitutive relationship between the fluid's (extra) stress and velocity. As the fluid's velocity, pressure, and stress (a symmetric tensor) each need to be determined, a direct approximation technique would require the solution of a very large nonlinear system of equations at each time step.

The fractional step θ-method [1–3] is an appealing numerical approximation technique for this problem for several reasons. The θ-method separates the updates for velocity/pressure and stress into several substeps. Variables are alternately lagged in the updates to reduce the size of the algebraic systems which have to be solved at each substep. In addition, the splitting allows the use of appropriate approximation techniques to resolve the resulting parabolic equations for velocity and pressure and the hyperbolic equation for stress. An additional benefit of the θ-method [3] is that the sequential nature of the velocity, pressure, and stress updates means that the algebraic systems in each substep are linear.

Research on viscoelastic materials can be traced back to Maxwell, Boltzmann, and Volterra, in the late eighteen hundreds, but it was the work of Oldroyd in 1950 that produced a constitutive model that worked well when modeling fluids with large deformations [4,5]. Since Oldroyd's original work, many constitutive equations have been formulated to describe the motion of viscoelastic fluids. These include the models of Giesekus [6], Oldroyd [7], and Phan-Thien and Tanner [8], as well as the Johnson and Segalman [9] constitutive model used in this work.

Error analysis of finite element approximations to steady state viscoelastic flow was first done by Baranger and Sandri in [10] using a discontinuous Galerkin (DG) formulation of the constitutive equation. In [11] Sandri presented the analysis of the steady state problem using a streamline upwind Petrov–Galerkin (SUPG) method of stabilization. The time-dependent problem was first analyzed by Baranger and Wardi in [12], using an implicit Euler temporal discretization and

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DG approximation for the hyperbolic constitutive equation. Ervin and Miles analyzed the problem using an implicit Euler time discretization and a SUPG discretization for the stress in [13]. The analysis of a modified Euler-SUPG approximation to the transient viscoelastic flow problem was presented by Bensaada and Esselaoui in [14]. The temporal accuracy of the approximation schemes studied in [12,14,13] are all $O(\Delta t)$. The work of Machmoum and Esselaoui in [15] examined time-dependent viscoelastic flow using a characteristics method that has accuracy $O((h^2/\sqrt{\Delta t})+\Delta t)$. Ervin and Heuer proposed a Crank–Nicolson time discretization method [16] which they showed was $O\left(\Delta t^2\right)$. Their method uses a three level scheme to approximate the nonlinear terms in the equations. Consequently their approximation algorithm only requires linear systems of equations to be solved. In [17] Bonito, Clément, and Picasso use an implicit function theorem to analyze a simplified time-dependent viscoelastic flow model where the convective terms were neglected.

The fractional step θ-method was introduced, and its temporal approximation accuracy studied, for a symmetric, positive definite spatial operator, by Glowinski and Périaux in [18]. The method is widely used for the accurate approximation of the Navier–Stokes equations (NSE) [19–21]. In [22], Klouček and Rys showed, assuming a unique solution existed, that the θ-method approximation converged to the solution of the NSE as the spatial and mesh parameters tend to zero (h, $\Delta t \to 0^+$). The temporal discretization error for the θ-method for the NSE was studied by Müller-Urbaniak in [23] and shown to be of second order.

The implementation of the fractional step θ-method in [3] for viscoelasticity differs significantly from that for the NSE. For the NSE at each substep the discretization contains the *stabilizing* operator $-\Delta\mathbf{u}$. For the viscoelasticity problem the middle substep, when resolving the stress, is a pure convection (transport) problem that requires stabilization in order to control the creation of spurious oscillations in the numerical approximation. Marchal and Crochet [24] were the first to use streamline upwinding to stabilize the hyperbolic constitutive equation in viscoelastic flow. A second common approach to stabilizing the convective transport problem is to use a discontinuous Galerkin (DG) approximation for the stress [25,10].

In [26] the authors showed that the fractional step θ-method for a linear convection–diffusion problem is second order accurate with respect to the temporal discretization. Similar to the viscoelastic model, the linear convection–diffusion equations are a coupled hyperbolic/parabolic system. The additive split in the θ-method allowed the distinct modeling equation phenomena (hyperbolic convective transport and parabolic diffusion) to be updated sequentially. A SUPG approximation technique was used to stabilize the resulting hyperbolic equation. Our θ-method work on the convection–diffusion equations was extended to viscoelastic fluid flow in [27] where a priori error estimates for a "Stokes-like problem", assuming known stress, and a constitutive model, assuming known velocity and pressure, were established. Here, we extend the work in [27] to obtain an a priori error estimate for the full θ-method applied to viscoelastic fluid flow.

The remainder of this document is organized as follows: in Section 2 the mathematical model and θ-method for viscoelastic fluid flow are introduced. Section 3 gives the mathematical notation needed in order to formulate the problem in an appropriate mathematical setting. The unique solvability and a priori error estimates for the θ-method applied to the viscoelastic modeling equations are presented in Section 4. Numerical computations confirming the theoretical results and demonstrating the θ-method are given in Section 5.

2. The mathematical model and θ-method approximation

In this section the modeling equations for viscoelastic fluid flow as well as a fractional step θ-method approximation scheme are presented.

The Johnson–Segalman model for viscoelastic fluid flow

The non-dimensional modeling equations for an inertialess (i.e. $\mathbf{u}\cdot\nabla\mathbf{u}\approx 0$) viscoelastic fluid in a given domain $\Omega\subset\mathbb{R}^{\acute{d}}$ ($\acute{d}=2,3$) using a Johnson–Segalman constitutive equation are written as:

$$\boldsymbol{\sigma}+\lambda\left(\frac{\partial\boldsymbol{\sigma}}{\partial t}+\mathbf{u}\cdot\nabla\boldsymbol{\sigma}+g_a(\boldsymbol{\sigma},\nabla\mathbf{u})\right)-2\alpha\mathbf{d}(\mathbf{u})=\mathbf{0}\quad\text{in }\Omega, \tag{2.1}$$

$$Re\frac{\partial\mathbf{u}}{\partial t}+\nabla p-2(1-\alpha)\nabla\cdot\mathbf{d}(\mathbf{u})-\nabla\cdot\boldsymbol{\sigma}=\mathbf{f}\quad\text{in }\Omega, \tag{2.2}$$

$$\nabla\cdot\mathbf{u}=0\quad\text{in }\Omega, \tag{2.3}$$

$$\mathbf{u}=\mathbf{0}\quad\text{on }\partial\Omega, \tag{2.4}$$

$$\mathbf{u}(0,x)=\mathbf{u}_0(x)\quad\text{in }\Omega, \tag{2.5}$$

$$\boldsymbol{\sigma}(0,x)=\boldsymbol{\sigma}_0(x)\quad\text{in }\Omega. \tag{2.6}$$

Here (2.1) is the constitutive equation relating the fluids velocity $\mathbf{u}$ to the stress $\boldsymbol{\sigma}$, and (2.2) and (2.3) are the conservation of momentum and conservation of mass equations. The fluid pressure is denoted by p. The Weissenberg number λ is a dimensionless constant defined as the product of a characteristic strain rate and the relaxation time of the fluid [28]. Note