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Sharp discrete inequalities and applications to discrete variational problems☆

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ABSTRACT

Some new discrete inequalities involving monotonic or convex functions are obtained. While these are interesting inequalities in their own right, they can be applied to solving certain types of discrete variational problems effectively.

0. Introduction

While it is well known that integral inequalities are of fundamental importance in the study of qualitative as well as quantitative properties of solutions of differential and integral equations, it is less commonly recognized that "sharp" integral inequalities, that is, integral inequalities whose bounds are actually achievable, can be applied to solve Calculus of Variations problems in a very effective manner. In fact, the classical treatment of the Calculus of Variations involves the consideration of the Euler–Lagrange equations (see, e.g. [1]) which can be very complicated and in many situations impossible to solve for explicit solutions, in which cases one would have to settle for conservation laws or first integrals, which is, in many cases, rather unsatisfactory. But with "sharp" integral inequalities, certain types of Calculus of Variations problems can be solved directly for the optimal solution without having to go through the classical steps of considering the Euler–Lagrange equations, nor to determining whether the solution satisfies the sufficient condition for an extremum (for details of this treatment, one is referred to [2,3]).

The present paper serves as a discretization of the aforesaid treatment of variational problems in [2,3]. We first establish some new sharp discrete inequalities involving functions satisfying certain types of monotonicity. These are themselves interesting inequalities in their own right. We then generalize these to sharp discrete inequalities involving convex functions which cover a wide range of examples. These inequalities are then applied to solve certain types of discrete variational problems directly for optimal solutions in a very effective manner. For the sake of simplicity, we only work on some relatively simple situations. But as the method of the treatment is systematic and rather algorithmic, one easily sees that the same technique also works for more complicated discrete variational problems.

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1. Some basic discrete inequalities

In this section, some sharp inequalities concerning discrete functions satisfying certain monotonicity are obtained. As can be seen in Lemma 1.1 and Theorem 1.3, in certain situations the monotonicity condition can be lifted. Furthermore, some results in this section will be generalized to more general settings in Section 2.

Throughout the paper, $M \geq 1$ is a natural number. For any non-negative integer k, $\mathbb{Z}_k$ denotes the lattice $\{0, 1, \ldots, k-1\} \subset \mathbb{R}$. We write $\mathbb{R}_+ := (0, \infty)$, $\mathbb{R}_0 := [0, \infty)$.

Lemma 1.1. *Let $F, G : [0, M-1] \to \mathbb{R}$ be any functions with F continuous. It is elementary to see that there is a point $c \in [0, M-1]$ such that*

$$F(c) = \frac{1}{M} \sum_{n=0}^{M-1} F(n).$$

Let $\overline{F}(x) := F(x) - F(c)$ and $\overline{G}(x) = G(x) - G(c)$.

(i) *If $\overline{F}\,\overline{G} \leq 0$ on $\mathbb{Z}_M$, then*

$$\Big(\sum_{n=0}^{M-1} F(n)\Big)\Big(\sum_{n=0}^{M-1} G(n)\Big) \geq M \sum_{n=0}^{M-1} F(n)G(n).$$

(ii) *If $\overline{F}\,\overline{G} \geq 0$ on $\mathbb{Z}_M$, then*

$$\Big(\sum_{n=0}^{M-1} F(n)\Big)\Big(\sum_{n=0}^{M-1} G(n)\Big) \leq M \sum_{n=0}^{M-1} F(n)G(n).$$

Furthermore, in either case the equality holds if and only if $\overline{F}\,\overline{G} = 0$ on $\mathbb{Z}_M$.

Proof. Since F is continuous and

$$\min_{[0,M-1]} F \leq \min_{\mathbb{Z}_M} F \leq \frac{1}{M} \sum_{n=0}^{M-1} F(n) \leq \max_{\mathbb{Z}_M} F \leq \max_{[0,M-1]} F,$$

by intermediate value theorem, there is a point $c \in [0, M-1]$ such that

$$F(c) = \frac{1}{M} \sum_{n=0}^{M-1} F(n).$$

Now if $\overline{F}\,\overline{G} \leq 0$ on $\mathbb{Z}_M$, then

$$\begin{aligned}
\Big(\sum_{n=0}^{M-1} F(n)\Big)\Big(\sum_{n=0}^{M-1} G(n)\Big) &= M \sum_{n=0}^{M-1} F(c)G(n) \\
&= M\Big[\sum_{n=0}^{M-1} F(n)G(n) - \sum_{n=0}^{M-1} \overline{F}(n)G(n)\Big] \\
&= M \sum_{n=0}^{M-1} F(n)G(n) - M\Big[\sum_{n=0}^{M-1} \overline{F}(n)\overline{G}(n) + \sum_{n=0}^{M-1} \overline{F}(n)G(c)\Big] \\
&= M \sum_{n=0}^{M-1} F(n)G(n) - M \sum_{n=0}^{M-1} \overline{F}(n)\overline{G}(n) \\
&\geq M \sum_{n=0}^{M-1} F(n)G(n)
\end{aligned}$$

and it is clear that the equality holds if and only if $\overline{F}\,\overline{G} = 0$ on $\mathbb{Z}_M$. This proves (i). The proof of (ii) is analogous. □

Theorem 1.2. *Let $f, g : \mathbb{Z}_M \to \mathbb{R}$ be functions such that f is monotonic increasing and g is monotonic decreasing. Then*

$$\Big(\sum_{n=0}^{M-1} f(n)\Big)\Big(\sum_{n=0}^{M-1} g(n)\Big) \geq M \sum_{n=0}^{M-1} f(n)g(n), \tag{1}$$

where the equality holds if and only if $f =$ constant or $g =$ constant.