Journal of Computational and Applied Mathematics 232 (2009) 187–200

Contents lists available at ScienceDirect

Journal of Computational and Applied Mathematics

journal homepage: www.elsevier.com/locate/cam

The existence of countably many positive solutions for some nonlinear nth order m-point boundary value problems ☆

Yude Ji *, Yanping Guo

College of Sciences, Hebei University of Science and Technology, Shijiazhuang, 050018, Hebei, PR China

ARTICLE INFO

Article history:
Received 13 October 2008

MSC:
34B15

Keywords:
Boundary value problem
Green's function
Krasnoselskii's fixed point theorem
Holder's inequality
Multiple positive solution

ABSTRACT

In this paper, we consider the existence of countably many positive solutions for nth-order m-point boundary value problems consisting of the equation

$$u^{(n)}(t)+a(t)f(u(t))=0,\quad t\in(0,1),$$

with one of the following boundary value conditions:

$$u(0)=\sum_{i=1}^{m-2}k_iu(\xi_i),\quad u'(0)=\cdots=u^{(n-2)}(0)=0,\quad u(1)=0,$$

and

$$u(0)=0,\quad u'(0)=\cdots=u^{(n-2)}(0)=0,\quad u(1)=\sum_{i=1}^{m-2}k_iu(\xi_i),$$

where $n\geq 2$, $k_i>0\,(i=1,2,\ldots,m-2)$, $0<\xi_1<\xi_2<\cdots<\xi_{m-2}<1$, $a(t)\in L^p[0,1]$ for some $p\geq 1$ and has countably many singularities in $[0,\frac{1}{2})$. The associated Green's function for the nth order m-point boundary value problem is first given, and we show that there exist countably many positive solutions using Holder's inequality and Krasnoselskii's fixed point theorem for operators on a cone.

1. Introduction

The existence of positive solutions for nonlinear second order multi-point boundary value problems have been studied by several authors. We refer the reader to [1–6] and references therein. Recently, the existence of positive solutions for high order multi-point boundary value problems has been studied by some authors. For details, see, for example, [7–9]. However, the high order multi-point boundary value problems treated in the above-mentioned references do not discuss problems with singularities. For the singular case of high order multi-point boundary value problems, to the author's knowledge, no one has studied the existence of positive solutions in the case. Very recently, Kaufmann and Kosmatov [10] showed that there exist countably many positive solutions for the two-point boundary value problems, with infinitely many singularities of following form:

$$\begin{cases}-u''(t)=a(t)f(u(t)), & 0<t<1,\\ u(0)=0, & u(1)=0,\end{cases}$$

where $a(t)\in L^p[0,1]$ for some $p\geq 1$ and has countably many singularities in $[0,\frac{1}{2})$.

☆ The project is supported by the Natural Science Foundation of Hebei Province (A2009000664) and the Foundation of Hebei University of Science and Technology (XL2006040).

* Corresponding author. Tel.: +86 13623307996.
E-mail address: jiyude-1980@163.com (Y. Ji).

0377-0427/$ – see front matter Crown Copyright © 2009 Published by Elsevier B.V. All rights reserved.
doi:10.1016/j.cam.2009.05.023

Motivated by the result of [10], in this paper we are interested in the existence of countably many positive solutions for nth-order m-point boundary value problems consisting of the equation

$$u^{(n)}(t) + a(t)f(u(t)) = 0, \quad t \in (0, 1), \tag{1.1}$$

with one of the following boundary value conditions:

$$u(0) = \sum_{i=1}^{m-2} k_i u(\xi_i), \qquad u'(0) = \cdots = u^{(n-2)}(0) = 0, \quad u(1) = 0, \tag{1.2}$$

and

$$u(0) = 0, \quad u'(0) = \cdots = u^{(n-2)}(0) = 0, \quad u(1) = \sum_{i=1}^{m-2} k_i u(\xi_i), \tag{1.3}$$

where $n \geq 2, k_i > 0\,(i = 1, 2, \ldots, m-2), 0 < \xi_1 < \xi_2 < \cdots < \xi_{m-2} < 1, f \in C([0, +\infty), [0, +\infty)), a(t) \in L^p[0, 1]$ for some $p \geq 1$ and has countably many singularities in $[0, \frac{1}{2})$. We show that the boundary value problems (1.1), (1.2) and (1.1), (1.3) have countably many solutions if a and f satisfy some suitable conditions. The key tool in our approach is the Holder's inequality and Krasnoselskii's fixed point theorem for operators on a cone.

We will suppose the following conditions are satisfied:

(H_1) there exists a sequence $\{t_k\}_{k=1}^{\infty}$ such that $t_{k+1} < t_k\ (k \in N), t_1 < \frac{1}{2}, \lim_{k\to\infty} t_k = t^* \geq 0$ and $\lim_{t\to t_k} a(t) = +\infty$ for all $k = 1, 2, \ldots$;

(H_2) there exists $H > 0$ such that $a(t) \geq H$ for all $t \in [t^*, 1 - t^*]$;

(H_3) there exists a $p \geq 1$ such that $a(t) \in L^p[0, 1]$;

(H_4) $f \in C([0, +\infty), [0, +\infty))$;

(H_5) $n \geq 2, k_i > 0\,(i = 1, 2, \ldots, m-2), 0 < \xi_1 < \xi_2 < \cdots < \xi_{m-2} < 1, 0 < \sum_{i=1}^{m-2} k_i(1 - \xi_i^{n-1}) < 1$;

(H_5') $n \geq 2, k_i > 0\,(i = 1, 2, \ldots, m-2), 0 < \xi_1 < \xi_2 < \cdots < \xi_{m-2} < 1, 0 < \sum_{i=1}^{m-2} k_i \xi_i^{n-1} < 1$.

We show that if $a(t)$ satisfies conditions (H_1)–(H_3) and if f satisfies oscillatory-like growth about a wedge, then the boundary value problem (1.1), (1.2) and (1.1), (1.3) have infinitely many solutions.

The paper is organized as follows. In Section 2, we provide some necessary background. In particular, we state a fixed point theorem due to Krasnoselskii's and Holder's inequality. In Section 3, the associated Green's function for the nth order two point boundary value problem is given and we also look at some properties of the Green's function associated with the boundary value problem. In Section 4, the associated Green's function for the nth order m-point boundary value problem is first given and we also look at some properties of the Green's function associated with the boundary value problem (1.1) and (1.2). We present the boundary value problems (1.1) and (1.2) have countably many solutions if a and f satisfy some suitable conditions. In Section 5, the associated Green's function for the nth order m-point boundary value problem is first given and we also look at some properties of the Green's function associated with the boundary value problem (1.1) and (1.3). We present the boundary value problems (1.1) and (1.3) have countably many solutions if a and f satisfy some suitable conditions. In Section 6, we present our main result as well as provide an example of a family of functions $a(t)$ that satisfy conditions (H_1)–(H_3) and two simple examples are presented to illustrate the applications of the obtained results.

2. Preliminary results

Definition 2.1. Let E be a Banach space over R. A nonempty convex closed set $K \subset E$ is said to be a cone, provided that

(i) $au \in K$ for all $u \in K$ and all $a \geq 0$;

(ii) $u, -u \in K$ implies $u = 0$.

Theorem 2.1 (*Krasnoselskii's Fixed Point Theorem*). *Let E be a Banach space and let $P \subset E$ be a cone. Assume Ω_1, Ω_2 are bounded open subsets of E such that $0 \in \Omega_1 \subset \overline{\Omega}_1 \subset \Omega_2$. Suppose that*

$$T : P \bigcap (\overline{\Omega}_2 \setminus \Omega_1) \to P$$

is a completely continuous operator such that either

(i) $\|Tu\| \leq \|u\|, u \in P \bigcap \partial\Omega_1$, *and* $\|Tu\| \geq \|u\|, u \in P \bigcap \partial\Omega_2$, *or*

(ii) $\|Tu\| \geq \|u\|, u \in P \bigcap \partial\Omega_1$, *and* $\|Tu\| \leq \|u\|, u \in P \bigcap \partial\Omega_2$.

Then T has a fixed point in $P \bigcap (\overline{\Omega}_2 \setminus \Omega_1)$.

In order to establish some of the norm inequalities in Theorem 2.1 we will need Holder's inequality. We use standard notation of $L^p[a, b]$ for the space of measurable functions such that

$$\int_0^1 |f(s)|^p \, ds < \infty,$$