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Shifted lattice rules based on a general weighted discrepancy for integrals over Euclidean space

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ABSTRACT

We approximate weighted integrals over Euclidean space by using shifted rank-1 lattice rules with good bounds on the "generalised weighted star discrepancy". This version of the discrepancy corresponds to the classic L_∞ weighted star discrepancy via a mapping to the unit cube. The weights here are general weights rather than the product weights considered in earlier works on integrals over $\mathbb{R}^d$. Known methods based on an averaging argument are used to show the existence of these lattice rules, while the component-by-component technique is used to construct the generating vector of these shifted lattice rules. We prove that the bound on the weighted star discrepancy considered here is of order $O(n^{-1+\delta})$ for any $\delta > 0$ and with the constant involved independent of the dimension. This convergence rate is better than the $O(n^{-1/2})$ achieved so far for both Monte Carlo and quasi-Monte Carlo methods.

1. Introduction

We consider integrals over Euclidean space given by

$$I_d(f, \rho) = \int_{\mathbb{R}^d} f(\mathbf{x})\rho(\mathbf{x})\,\mathrm{d}\mathbf{x}, \tag{1}$$

where $\rho(\mathbf{x})$ is a probability density function. Hence $\rho(\mathbf{x}) \geq 0$ for any $\mathbf{x} \in \mathbb{R}^d$ and $\int_{\mathbb{R}^d} \rho(\mathbf{x})\,\mathrm{d}\mathbf{x} = 1$. We also assume that the probability density $\rho(\mathbf{x})$ has the product form

$$\rho(\mathbf{x}) = \prod_{j=1}^{d} \rho_j(x_j),$$

where each ρ_j is a probability density over $\mathbb{R}$. For simplicity we shall assume that all the densities ρ_j are equal, however the results can be extended in the case when these densities are different for each coordinate direction.

Integrals over unbounded regions may be studied by first employing a mapping to the unit cube (see [1–3]) and then constructing a shifted lattice rule over the unit cube (see [4,5]). In the 1-dimensional case, we can use the following transform:

$$u = \Phi(x) = \int_{-\infty}^{x} \rho(t)\,\mathrm{d}t, \quad \forall x \in \mathbb{R}. \tag{2}$$

The inverse mapping will be $\Phi^{-1} : (0, 1) \to \mathbb{R}$, $\Phi^{-1}(u) = x$. In the d-dimensional case, the mapping (2) will be applied to each coordinate direction. So, if we take $\mathbf{x} = (x_1, x_2, \ldots, x_d) \in \mathbb{R}^d$, then $\Phi(\mathbf{x}) = (\Phi(x_1), \Phi(x_2), \ldots, \Phi(x_d))$. In the same

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manner, the inverse mapping will also be applied component-wise. The integral (1) will thus become

$$I_d(f,\rho)=\int_{[0,1]^d} f(\Phi^{-1}(\mathbf{u}))\,\mathrm{d}\mathbf{u}=\int_{[0,1]^d} g(\mathbf{u})\,\mathrm{d}\mathbf{u}:=I_d(g),$$

where $g=f\circ\Phi^{-1}$, with the composition of f with Φ^{-1} applied for each coordinate direction. Integrals over the unit cube might be approximated by quadrature rules of the form

$$Q_{n,d}(g)=\frac{1}{n}\sum_{k=0}^{n-1} g(\mathbf{w}_k)=\frac{1}{n}\sum_{k=0}^{n-1} f(\mathbf{t}_k),$$

where $\mathbf{w}_k\in[0,1]^d$, for all $0\le k\le n-1$ and $\mathbf{t}_k=\Phi^{-1}(\mathbf{w}_k)\in\mathbb{R}^d$ for all $0\le k\le n-1$ with the inverse mapping Φ^{-1} applied component-wise.

In this paper we are interested in constructing shifted rank-1 lattice rules suitable for integrals of the form (1) by using a weighted star discrepancy as a criterion of goodness. It is common that mappings of the form (2) lead to integrands that are unbounded near the boundary of the unit cube, and usually shifted lattice rules are employed in order to avoid evaluation at the singularities. Such shifted rank-1 lattice rules are quadrature rules of the form

$$Q_{n,d}(g)=\frac{1}{n}\sum_{k=0}^{n-1} g\left(\left\{\frac{k\mathbf{z}}{n}+\boldsymbol{\Delta}\right\}\right), \tag{3}$$

where $\mathbf{z}$ is the generating vector having all the components assumed to be relatively prime with n and $\boldsymbol{\Delta}\in[0,1)^d$ is the shift. Shifted lattice rules suitable for integrals over unbounded regions have been previously constructed in [4,5]. However in both of these papers it was assumed that the weights have a "product" form (see the next section for details on weight settings). The purpose of the present paper is to construct shifted lattice rules for integrals of the form (1) in a "general weighted" setting. Such a weight setting was used in [6], where rank-1 lattice rules having a low weighted star discrepancy were constructed and it was also pointed out that the techniques therein could be used for weighted integrands over unbounded regions, but without effectively presenting such a construction. Since here we make the same assumptions over the weights as in [6], this paper is intended to extend the results of [6].

In [4,5], the resulting error had the order of magnitude of $O(n^{-1/2})$, which is also the typical convergence expected from a Monte Carlo method. In both of these papers the authors remarked by using numerical experiments that their lattice rules perform significantly better than Monte Carlo methods, however a mathematical proof of a better convergence than $O(n^{-1/2})$ was not given. As we shall see later, the weighted star discrepancy used here in order to assess the goodness of a shifted lattice rule of the form (3) will have a better convergence order than the convergence observed in [4,5]. This convergence is the optimal $O(n^{-1+\delta})$ for any $\delta>0$ and with the involved constant independent of the dimension. Such an optimal convergence rate has been also obtained for instance in [3,6,7]. We remark that in [3], the authors used a similar discrepancy as the discrepancy defined below by (5), but under a product weighted assumption and without providing an explicit construction of the quadrature points. In the present paper, we give an explicit construction and moreover, we allow the weights to have more generality than the product weights used in [3]. We also remark that the techniques used here resemble with the typical techniques used in reproducing kernel Hilbert spaces (used for instance in [4,5]), although there is no reproducing kernel.

Let us also remark that under a general weighted assumption, there are no results to date in the specialised literature regarding construction of lattice rules suitable for integrands over unbounded regions, so we also fill a gap in this sense. Moreover, in the last section we give an extension of the results from [8]. Such an extension is important at the analysis of the computational costs of the construction algorithm and could be used in further work that employs a similar form of the quadrature points as here.

2. Generalised weighted star discrepancy

We first mention that the local star discrepancy of a set P_n of n points in the unit cube is defined by

$$\mathrm{discr}(\mathbf{x},P_n):=\frac{|[\mathbf{0},\mathbf{x})\cap P_n|}{n}-\prod_{j=1}^{d}x_j,$$

where $\mathbf{x}=(x_1,x_2,\ldots,x_d)\in[0,1]^d$. Let now $\mathfrak{u}$ be an arbitrary non-empty subset of $\mathcal{D}:=\{1,2,\ldots,d-1,d\}$ and let $\boldsymbol{\gamma}_\mathfrak{u}$ be a non-negative weight associated with $\mathfrak{u}$. For the vector $\mathbf{x}\in[0,1]^d$, let $\mathbf{x}_\mathfrak{u}$ denote the vector from $[0,1]^{|\mathfrak{u}|}$ containing the components of $\mathbf{x}$ whose indices belong to $\mathfrak{u}$. By $(\mathbf{x}_\mathfrak{u},\mathbf{1})$ we mean the vector from $[0,1]^d$ whose jth component is x_j if $j\in\mathfrak{u}$ and 1 if $j\notin\mathfrak{u}$. The "classical" weighted star discrepancy from [6] was defined by

$$D^*_{n,\boldsymbol{\gamma}}(P_n):=\max_{\mathfrak{u}\subseteq\mathcal{D}}\boldsymbol{\gamma}_\mathfrak{u}\sup_{\mathbf{x}_\mathfrak{u}\in[0,1]^{|\mathfrak{u}|}}\left|\mathrm{discr}((\mathbf{x}_\mathfrak{u},\mathbf{1}),P_n)\right|. \tag{4}$$