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Splitting-based conjugate gradient method for a multi-dimensional linear inverse heat conduction problem

Dinh Nho Hào [a,*], Nguyen Trung Thành [b], Hichem Sahli [b]

[a] Hanoi Institute of Mathematics, 18 Hoang Quoc Viet Road, 10307 Hanoi, Viet Nam

[b] Vrije Universiteit Brussel, Department of Electronics and Informatics, Pleinlaan 2, 1050 Brussels, Belgium

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ABSTRACT

In this paper we consider a multi-dimensional inverse heat conduction problem with time-dependent coefficients in a box, which is well-known to be severely ill-posed, by a variational method. The gradient of the functional to be minimized is obtained by the aid of an adjoint problem, and the conjugate gradient method with a stopping rule is then applied to this ill-posed optimization problem. To enhance the stability and the accuracy of the numerical solution to the problem, we apply this scheme to the discretized inverse problem rather than to the continuous one. The difficulties with large dimensions of discretized problems are overcome by a splitting method which only requires the solution of easy-to-solve one-dimensional problems. The numerical results provided by our method are very good and the techniques seem to be very promising.

1. Introduction

Inverse heat conduction problems (IHCPs), because of their important applications in many branches of technology, science, etc., have been extensively studied over the last 50 years or so. Although there exists a vast literature on one-dimensional problems, there are much fewer papers devoted to multi-dimensional cases, especially when the coefficients of the equations describing the heat transfer processes depend on time. For recent surveys on the subject we refer to [1–3,6,9]. The aim of this work is to suggest a fast and stable numerical method for a multi-dimensional IHCP with time-dependent coefficients in a parallelepiped. To our knowledge, our result is one of very few papers dealing with multi-dimensional IHCPs with time-dependent coefficients.

Let Ω be the open parallelepiped $(l_1, L_1) \times (l_2, L_2) \times \cdots \times (l_n, L_n)$ with $l_1, l_2, \ldots, l_n, L_1, L_2, \ldots, L_n (n \geq 2)$ being given. Denote by $\partial\Omega$ the boundary of Ω. For $t \in (0, T]$, set $Q_t := \Omega \times (0, t]$, $S_t := \partial\Omega \times (0, t]$, $S = S_T$. Suppose that $\partial\Omega$ is split into three parts Γ_1, Γ_2 and Γ_3, where $\Gamma_i \cap \Gamma_j = \emptyset$, $i, j = 1, 2, 3$, $i \neq j$. We denote $\Gamma_i \times (0, T]$ by S_i, $i = 1, 2, 3$. Consider the problem of determining $\partial u/\partial N|_{S_3}$ and $u|_{t=0}$ from the system

$$\frac{\partial u}{\partial t} - \sum_{i=1}^{n} \frac{\partial}{\partial x_i}\left(a_i(x, t)\frac{\partial u}{\partial x_i}\right) + a(x, t)u = f, \quad (x, t) \in Q_T, \tag{1.1}$$

$$u|_{S_1} = \varphi(\xi, t), \quad (\xi, t) \in S_1, \tag{1.2}$$

$$\frac{\partial u}{\partial N}|_{S_1 \cup S_2} = g(\xi, t), \quad (\xi, t) \in S_1 \cup S_2. \tag{1.3}$$

* Corresponding author.
E-mail addresses: hao@math.ac.vn, hao@etro.vub.ac.be (D.N. Hào), ntthanh@etro.vub.ac.be (N.T. Thành), hsahli@etro.vub.ac.be (H. Sahli).

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Here, φ and g are given functions, ν is the outer normal to S, and

$$\frac{\partial u}{\partial N}|_S := \sum_{i=1}^{n} a_i(x,t)u_{x_i}\cos(\nu, x_i)|_S.$$

In this paper, we assume that the following conditions are satisfied

$$a_i,\ a \in C(\bar{Q}_T), \tag{1.4}$$

$$a_i(x,t) \geq \lambda > 0, \quad a(x,t) \geq 0, \forall (x,t) \in \bar{Q}_T, \forall i = 1, 2, \ldots, n, \tag{1.5}$$

$$f \in L_2(Q_T), \qquad \varphi \in L_2(S_1), \qquad g \in L_2(S_1 \cup S_2), \tag{1.6}$$

The problem (1.1)–(1.3) is severely ill-posed (see, e.g. [1–3,6]). In this paper we shall use the variational method suggested in [5] and the conjugate gradient method (CGM) to this IHCP. The idea is very simple: since the initial condition and the Neumann condition $\partial u/\partial N|_{S_3}$ are not known, we consider them as a control v to minimize the defect functional $J_0(v) = 1/2\|u|_{S_1} - \varphi\|^2_{L_2(S_1)}$. The gradient of the defect functional is found via the direct and adjoint problems. Since the optimization problem is still unstable, we have to use a regularization method for it. In fact, we shall use the CGM with a stopping rule proposed by Nemirovskii [13] which has been proved to have optimal order regularization properties. It then comes out that for evaluating the gradient, one should first numerically solve the direct and adjoint problems and thus obtain an approximation to the gradient. However, it should be noted that when we discretize the variational problem we get a discretized functional, and the gradient of this new one is not the same as that obtained by the above method. In fact, the last is only an approximation of it, and the approximation error becomes more and more significant in the iterative procedure. Besides, since direct discretization of the direct and adjoint problems leads to extremely large systems of algebraic equations, their numerical solutions are extremely expensive. To overcome these difficulties we suggest the following scheme: (1) discretize the direct problem and form a corresponding discretized functional, (2) introduce the discretized adjoint problem and (3) evaluate the gradient of the discretized functional, (4) use the CGM for the discretized variational problem. To avoid the large dimensions of the discretized problems we use a splitting method (see, e.g. [11, 18]) for this purpose. The technique only requires solving one-dimensional problems, the numerical solution of which is easily and directly calculated. The numerical results provided by our method are very good and the techniques seem to be very promising. Also, we make use of the Tikhonov regularization to the problem in combination with CGM. We found that numerical results with or without Tikhonov regularization are of the same quality. However, the algorithm converges faster without Tikhonov regularization.

The paper is structured as follows. Section 2 summarizes the direct and inverse problems as well as the CGM. In Section 3 we formulate the discretized direct and inverse problem and calculate the gradient of the discretized objective function. Some numerical examples are shown in Section 4 to illustrate the performance of the considered algorithm. Finally, some conclusions are drawn in Section 5.

The results of this paper have been reported at the 6th International Conference on Inverse Problems in Engineering: Theory and Practice, 15–19 June 2008, Dourdan (Paris), France [7].

2. Problem setting and conjugate gradient method

In this section we summarize some results on the inverse problem (1.1)–(1.3) and its related direct problem. For more details, we refer the reader to [6].

2.1. The direct problem

This part is devoted to a non-homogeneous second boundary value problem for linear parabolic equations. This problem is referred to as the direct one.

For the further discussions, we need the following definitions of Sobolev spaces [10]:

The space $H^1(\Omega)$ consists of all elements $u \in L_2(\Omega)$ having generalized derivatives u_{x_i} in $L_2(\Omega)$. The scalar product in $H^1(\Omega)$ is defined by

$$(u,v)_{H^1(\Omega)} = \int_\Omega \left(uv + \sum_{i=1}^{n} u_{x_i} v_{x_i} \right) \mathrm{d}x.$$

The space $H^{1,0}(Q_T)$ is the set of all elements $u \in L_2(Q_T)$ having generalized derivatives u_{x_i} in $L_2(Q_T)$ with the scalar product

$$(u,v)_{H^{1,0}(Q_T)} = \int_{Q_T} \left(uv + \sum_{i=1}^{n} u_{x_i} v_{x_i} \right) \mathrm{d}x\mathrm{d}t.$$

The space $H^{1,1}(Q_T)$ consists of all elements $u \in L_2(Q_T)$ having generalized derivatives u_{x_i}, u_t in $L_2(Q_T)$. Set

$$V^{1,0}(Q_T) := C([0,T]; L_2(\Omega)) \cap L_2((0,T); H^1(\Omega)).$$