Journal of Computational and Applied Mathematics 232 (2009) 523–538

Contents lists available at ScienceDirect

Journal of Computational and Applied Mathematics

journal homepage: www.elsevier.com/locate/cam

# Nyström method for Cauchy singular integral equations with negative index ☆

M.C. De Bonis *, C. Laurita

*Department of Mathematics, University of Basilicata, Via dell'Ateneo Lucano 10, 85100 Potenza, Italy*

ARTICLE INFO

*Article history:*
Received 17 September 2008
Received in revised form 4 February 2009

*MSC:*
65R20
45E05
41A05
41A10

*Keywords:*
Cauchy singular integral equations
Fredholm integral equations
Nyström method
Gaussian quadrature rule
Lagrange interpolation
Condition number

ABSTRACT

In this paper, the authors propose a Nyström method to approximate the solutions of Cauchy singular integral equations with constant coefficients having a negative index. They consider the equations in spaces of continuous functions with weighted uniform norm. They prove the stability and the convergence of the method and show some numerical tests that confirm the error estimates.



## 1. Introduction

Let us consider the following Cauchy singular integral equations with constant coefficients

$$af(y)v^{\alpha,\beta}(y) + \frac{b}{\pi}\int_{-1}^{1}\frac{f(x)}{x-y}v^{\alpha,\beta}(x)\mathrm{d}x + \mu\int_{-1}^{1}k(x,y)f(x)v^{\alpha,\beta}(x)\mathrm{d}x = g(y), \tag{1}$$

where $|y| < 1$, $a, b \in \mathbb{R}$ are constants such that $a^2 + b^2 = 1$, $b \neq 0$, $\mu \in \mathbb{R}$ and $k$ and $g$ are given functions on $(-1, 1)^2$ and $(-1, 1)$, respectively. The function $f$ is the unknown of the equation and $v^{\alpha,\beta}(x) = (1-x)^{\alpha}(1+x)^{\beta}$ is a Jacobi weight whose exponents $-1 < \alpha, \beta < 1$ are given by

$$\alpha = M - \frac{1}{2\pi \mathrm{i}}\log\left(\frac{a+\mathrm{i}b}{a-\mathrm{i}b}\right), \qquad \beta = N + \frac{1}{2\pi \mathrm{i}}\log\left(\frac{a+\mathrm{i}b}{a-\mathrm{i}b}\right),$$

with $M$ and $N$ integers chosen so that $\chi = -(\alpha+\beta) = -(M+N)$ is restricted to $\chi \in \{-1, 0, 1\}$.

This kind of integral equation appears in several problems of applied sciences and a wide literature on this topic is available. Among them, we mention the fundamental books and papers [1–12] and the references therein. Recently, in [13], such class of equations with index $\chi \in \{0, 1\}$ were considered in spaces of continuous functions with uniform norm. Using the regularization method [8,10,11], Fredholm equations are obtained and they are solved using projection-type methods.

☆ Supported by Italian Ministero dell'Università e della Ricerca, PRIN 2006 "Numerical methods for structured linear algebra and applications".

* Corresponding author.
*E-mail addresses:* mariacarmela.debonis@unibas.it (M.C. De Bonis), concetta.laurita@unibas.it (C. Laurita).

0377-0427/$ – see front matter © 2009 Elsevier B.V. All rights reserved.
doi:10.1016/j.cam.2009.06.028

In this paper, we consider equations of the form (1) with index $\chi = -1$ in spaces of continuous functions with uniform norm. The numerical treatment of Cauchy singular integral equations in case of negative index has not been so widely developed in literature ([14,6] see also [5,12]). The linear system derived from direct methods (for instance quadrature, collocation or discrete collocation) is overdetermined and, then, it becomes necessary to exclude one of the equations. However, the criteria used in choosing such an equation are not clear.

In the sequel, in order to fix the ideas, we shall consider Cauchy singular integral equations of the following type

$$(Df)(y) + \mu(Kf)(y) = g(y), \quad |y| < 1, \tag{2}$$

where $\mu \in \mathbb{R}$ and the operators $D$ and $K$ are defined as

$$(Df)(y) = \cos(\pi\alpha)f(y)v^{\alpha,1-\alpha}(y) - \frac{\sin(\pi\alpha)}{\pi}\int_{-1}^{1}\frac{f(x)}{x-y}v^{\alpha,1-\alpha}(x)\mathrm{d}x, \tag{3}$$

$$(Kf)(y) = \int_{-1}^{1} k(x,y)f(x)v^{\alpha,1-\alpha}(x)\mathrm{d}x, \tag{4}$$

respectively, with $\alpha \in (0, 1)$. In other words, we take the constant coefficients $a$ and $b$ appearing in (1) as $a = \cos(\pi\alpha)$ and $b = -\sin(\pi\alpha)$, respectively. The other possible choice is $a = -\cos\pi\alpha$, $b = \sin\pi\alpha$ and, in this case, what follows can be repeated word by word.

Following [13,15], the procedure we propose here consists in reducing (1), under suitable assumptions on $k$ and $g$, to an equivalent regularized Fredholm integral equation and in solving the latter by a Nyström-type method. This approach, based on some mapping properties of the dominant operator $D$ related to Eq. (1) (see Section 2.2), permits the solution of a determined and well conditioned linear system.

This paper is organized as follows. In Section 2 we introduce some function spaces and show some mapping properties of the operators $D$ and $K$. In Section 3 we describe the numerical method, and in Section 4 we focus on the computational aspects. In Section 5 we prove the results stated in Sections 3 and 4. Finally, in Section 6 we give some numerical tests.

## 2. Preliminaries

In this paper, $\mathcal{C}$ will denote a positive constant which may have different values in different formulas. We will write $\mathcal{C} \neq \mathcal{C}(a, b, \ldots)$ to say that $\mathcal{C}$ is independent of the parameters $a, b, \ldots$. If $A, B \geq 0$ are quantities depending on some parameters, we will write $A \sim B$, if there exists a positive constant $\mathcal{C}$ independent of the parameters of $A$ and $B$, such that $\frac{B}{\mathcal{C}} \leq A \leq \mathcal{C}B$.

### 2.1. Functional spaces

We are going to consider the integral equations (2) in the space

$$C_v = \left\{ f \in C^0((-1,1)) : \lim_{|x|\to 1}(fv)(x) = 0 \right\},$$

where $C^0(A)$ is the collection of the continuous functions in $A \subset [-1, 1]$ and $v(x) := v^{\rho,\theta}(x) = (1-x)^{\rho}(1+x)^{\theta}$, $\rho, \theta \geq 0$, is a Jacobi weight.

In case $\rho = 0$ (respectively, $\theta = 0$) $C_v$ consists of all continuous functions on $(-1, 1]$ (respectively, $[-1, 1)$) such that

$$\lim_{x\to -1}(fv)(x) = 0 \quad \left(\text{respectively, } \lim_{x\to 1}(fv)(x) = 0\right).$$

In the case where $\rho = \theta = 0$, we set $C_v = C^0([-1, 1])$. The space $C_v$ equipped with the norm

$$\|f\|_{C_v} := \|fv\| := \max_{|x|\leq 1}|(fv)(x)|$$

is complete. Sometimes, for brevity of notations, we shall write $\|f\|_A := \max_{x\in A}|f(x)|$, for $A \subseteq [-1, 1]$.

In the sequel we will also consider functions belonging to the Zygmund space $Z_r(v)$ defined as follows

$$Z_r(v) = \left\{ f \in C_v : \sup_{t>0}\frac{\Omega_{\varphi}^{k}(f,t)_v}{t^r} < +\infty,\ k > r > 0 \right\}, \quad r \in \mathbb{R}^+, k \in \mathbb{N},$$

by means of the main part of the modulus of continuity [16]

$$\Omega_{\varphi}^{k}(f,t)_v = \sup_{0<h\leq t}\|(\Delta_{h\varphi}^{k}f)v\|_{I_{h,k}},$$