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Schwarz methods for inequalities with contraction operators

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Abstract

We prove the convergence of some multiplicative and additive Schwarz methods for inequalities which contain contraction operators. The problem is stated in a reflexive Banach space and it generalizes the well-known fixed-point problem in the Hilbert spaces. Error estimation theorems are given for three multiplicative algorithms and two additive algorithms. We show that these algorithms are in fact Schwarz methods if the subspaces are associated with a decomposition of the domain. Also, for the one- and two-level methods in the finite element spaces, we write the convergence rates as functions of the overlapping and mesh parameters. They are similar with the convergence rates of these methods for linear problems. Besides the direct use of the five algorithms for the inequalities with contraction operators, we can use the above results to obtain the convergence rate of the Schwarz method for other types of inequalities or nonlinear equations. In this way, we prove the convergence and estimate the error of the one- and two-level Schwarz methods for some inequalities in Hilbert spaces which are not of the variational type, and also, for the Navier–Stokes problem. Finally, we give conditions of existence and uniqueness of the solution for all problems we consider. We point out that these conditions and the convergence conditions of the proposed algorithms are of the same type.

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1. Introduction

Literature on the domain decomposition methods is very large, and it is motivated by their capability in providing efficient algorithms for large scale problems. We can see, for instance, the papers in the proceedings of the annual conferences on domain decomposition methods starting in 1987 with [10] or those cited in the books [15,25,26,30]. Naturally, most of the papers dealing with these methods are dedicated to the linear elliptic problems. For the variational inequalities, the convergence proofs refer in general to the inequalities coming from the minimization of quadratic functionals. Also, most of the papers consider the convex set decomposed according to the space decomposition as a sum of convex subsets. To our knowledge very few papers deal with the application of these methods to nonlinear problems. We can cite in this direction the papers written by Boglaev [6], Dryja and Hackbusch [8], Lui [20–22], Tai and Espedal [27], and Tai and Xu [28], for nonlinear equations, Hoffmann and Zou [12], Zeng and Zhou [31], for inequalities having nonlinear source terms, and Badea [2], for the minimization of non-quadratic functionals.

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The multilevel or multigrid methods can be viewed as domain decomposition methods and we can cite the results obtained by Kornhuber [13–15], Mandel [23,24], Smith et al. [26], Tarvainen [29], Badea et al. [5], and Badea [3]. Evidently, this list is not exhaustive and it can be completed with other papers.

In this paper, we mainly deal with the convergence of the multiplicative and additive Schwarz methods for inequalities containing contraction operators. In Section 2, we introduce the framework of the paper. The problem is stated in a reflexive Banach space and it generalizes the well-known fixed-point problem in the Hilbert spaces. Section 3 is dedicated to some general subspace correction methods for the problem in previous section. We propose here three multiplicative algorithms and two additive algorithms, and give error estimation theorems for them. At the end of this section, we show that the given algorithms are in fact Schwarz methods if the used subspaces are associated with a decomposition of the domain. The general convergence theorems are based on some assumptions we have introduced in Section 2. The convergence rate essentially depends on a constant C_0 in these assumptions. For the one- and two-level methods, we are able to write this constant as a function of the overlapping and mesh parameters, and the convergence rates we get are similar with the convergence rate for the linear problems. This section generalizes the results in [20], where, using the proof technique introduced in [17–19], it is proved that the Schwarz method with two subdomains converges for fixed-point problems in a Hilbert space. Besides the direct use of the five algorithms for the solution of inequalities with contraction operators, we can use the previous results to obtain the convergence of the Schwarz methods for other types of inequalities or nonlinear equations. In Section 4, we give convergence theorems and estimate the error of the multiplicative and additive Schwarz methods (as well as for the one- and two-level methods) for inequalities in Hilbert spaces which do not come from the minimization of a functional and do not contain other terms which could help the convergence process (like contraction operators, for instance). We use here the preconditioned Richardson iteration associated to our problem. In this section as well as in Sections 2 and 5, we give some existence and uniqueness propositions for the solution of the problem we consider. We point out that the existence and uniqueness conditions in these propositions and the corresponding conditions in the convergence theorems of the proposed algorithms are of the same type. Naturally, the convergence condition is stronger. Evidently, all results concerning the inequalities are also valid for equations. In Section 5, we prove the convergence of the one- and two-level Schwarz methods for the Navier–Stokes problem. As we already said, our result shows that these methods converge if the viscosity of the fluid is large enough, and this condition is of the same type with the well-known existence and uniqueness condition of the solution.

2. General framework

Let V be a reflexive Banach space, $V_1, \ldots, V_m$, be some closed subspaces of V, and $K \subset V$ be a non-empty closed convex set. As we already said, we prove the convergence of some subspace correction algorithms which will be multiplicative or additive Schwarz methods in the case of the Sobolev spaces. For the multiplicative algorithms, we make the following:

Assumption 2.1. There exists a constant $C_0 > 0$ such that for any $w, v \in K$ and $w_i \in V_i$ with $w + \sum_{j=1}^{i} w_j \in K$, $i = 1, \ldots, m$, there exist $v_i \in V_i$, $i = 1, \ldots, m$, satisfying

$$w + \sum_{j=1}^{i-1} w_j + v_i \in K \quad \text{for } i = 1, \ldots, m, \tag{2.1}$$

$$v - w = \sum_{i=1}^{m} v_i, \tag{2.2}$$

and

$$\sum_{i=1}^{m} \|v_i\| \leqslant C_0 \left(\|v - w\| + \sum_{i=1}^{m} \|w_i\| \right). \tag{2.3}$$

This assumption looks complicated enough, but as we shall see in the following, it is satisfied for a large kind of convex sets in the Sobolev spaces. It has been introduced in a slightly modified form in [1], and then used in the present