Available online at www.sciencedirect.com
ScienceDirect
Journal of Computational and Applied Mathematics 222 (2008) 3–16
JOURNAL OF COMPUTATIONAL AND APPLIED MATHEMATICS
www.elsevier.com/locate/cam

Penalty methods for the numerical solution of American multi-asset option problems

Bjørn Fredrik Nielsen*, Ola Skavhaug, Aslak Tveito

Simula Research Laboratory, P.O. Box 134, N-1325 Lysaker, Norway
Department of Informatics, University of Oslo, P.O. Box 1080, Blindern, N-0316 Oslo, Norway

Received 27 February 2006; received in revised form 31 March 2007

Abstract

We derive and analyze a penalty method for solving American multi-asset option problems. A small, non-linear penalty term is added to the Black–Scholes equation. This approach gives a fixed solution domain, removing the free and moving boundary imposed by the early exercise feature of the contract. Explicit, implicit and semi-implicit finite difference schemes are derived, and in the case of independent assets, we prove that the approximate option prices satisfy some basic properties of the American option problem. Several numerical experiments are carried out in order to investigate the performance of the schemes. We give examples indicating that our results are sharp. Finally, the experiments indicate that in the case of correlated underlying assets, the same properties are valid as in the independent case.

MSC: 65M06; 65M12; 65Y20; 91B28

Keywords: American options; Penalty methods; Numerical solution

1. Introduction

American derivatives are popular trading instruments in present-day financial markets. We consider American put options where the payoff depends on more than one underlying asset. Such option prices can be modeled by higher dimensional generalizations of the original Black–Scholes equation [1]. The purpose of this paper is to extend the penalty method discussed in [19] to multi-asset American put option problems.

Various numerical techniques can be applied to price multi-variate derivatives. Higher dimensional generalizations of lattice binomial methods can be used, cf. [2], where European options based on three underlying options are solved numerically. Another way of pricing multi-asset derivatives is by the Monte Carlo simulation techniques, cf. [12]. In a wide range of scientific fields, finite element and finite difference methods (FEM and FDM) are popular. For studies of FEM and FDM concerning the numerical valuation of financial derivatives see, e.g., [11,27,6,26,4,5].

The idea behind the penalty method for multi-asset option models is similar to the method described in [19]. American put options can be exercised at any time before expiry. This introduces a free and moving boundary

* Corresponding author at: Simula Research Laboratory, P.O. Box 134, N-1325 Lysaker, Norway.
E-mail addresses: bjornn@simula.no (B.F. Nielsen), skavhaug@simula.no (O. Skavhaug), aslak@simula.no (A. Tveito).

0377-0427/$ - see front matter
doi:10.1016/j.cam.2007.10.041

problem. By adding a certain penalty term to the Black–Scholes equation, we extend the solution to a fixed domain. Furthermore, this term forces the solution to stay above the payoff function at expiry. Throughout the last decade, a number of papers addressing penalty schemes for American options have been published, see [10,15,9,17,7] and references therein.

The number of spatial degrees of freedom in the Black–Scholes equation equals the number of underlying assets involved in the contract. This means that the spatial dimension can be of order $O(10)$ (or even higher). Furthermore, as will be explained below, in order to solve an n-dimensional option problem one must typically solve a series of Black–Scholes equations with spatial dimensions $n-1, n-2, \ldots, 1$, leading to a very CPU demanding procedure. Consequently, it is necessary to design efficient numerical schemes for such problems. Stable higher order methods for the Black and Scholes equation have been introduced by Voss et al. [23] and Khaliq et al. [15]. Mesh-free methods based on radial basis functions may also reduce the computational efforts significantly, see Fasshauer et al. [9].

The present study is motivated by the scheme introduced by Forsyth and Vetzal in [28] for American options with stochastic volatility. In their work they add a source term to the discrete equations. Our method represents a refinement of their work in the sense that the penalty term is added to the continuous equation. For independent underlying assets, this leads to restrictions regarding the magnitude of the penalty term as well as conditions for the discretization parameters. Also, by choosing a semi-implicit finite difference discretization, we avoid solving non-linear algebraic equations and thereby enhance the overall computational efficiency.

We present numerical experiments illustrating the properties of the schemes. In the case of correlated underlying assets, we have been unable to derive proper bounds on the numerical solutions. However, numerical experiments indicate that similar properties are present in such cases.

This paper is organized as follows: In Section 2 we describe the multi-asset Black–Scholes equation, along with the penalty formulation of the problem. The boundary conditions corresponding to zero values of the underlying assets are obtained by solving lower dimensional Black–Scholes equations. In Section 4, numerical schemes for the two-factor model problem are derived, starting with specifying the two-factor model problem. First, an explicit scheme is presented, and then both a semi-implicit and a fully implicit scheme are defined. Analysis of these schemes are carried out in Section 4, under the assumption that the underlying assets are independent. Restrictions regarding the time step size and the penalty term are then provided for all three schemes. In the last section of this paper, we present a series of numerical experiments, starting with comparing the fully implicit and the semi-implicit schemes with respect to computational efficiency. In Section 5, we show that the numerical experiments indicate that for our model data, the restrictions derived in Section 4 for independent assets are also valid when the underlying assets are correlated. Finally, we make some conclusive remarks in Section 6.

2. American multi-asset option problems

The multi-dimensional version of the Black–Scholes equation takes the form

$$\frac{\partial P}{\partial t} + \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}\rho_{i,j}\sigma_i\sigma_j S_i S_j \frac{\partial^2 P}{\partial S_i \partial S_j} + \sum_{i=1}^{n}(r - D_i)S_i \frac{\partial P}{\partial S_i} - rP = 0, \tag{1}$$

see e.g. [8,16] or [24]. Here, P is the value of the contract, S_i is the value of the ith underlying asset, n is the number of underlying assets, $\rho_{i,j}$ is the correlation between asset i and asset j, r is the risk-free interest rate and D_i is the dividend yield paid by the ith asset.

The value of an American option at the time T of expiry of the contract is readily known as a function of the underlying assets. That is, $P(S_1, S_2, \ldots, S_n, T)$ is known and we want to use (1) to compute P throughout the time interval $[0, T]$. This means that $P(S_1, S_2, \ldots, S_n, T)$ provides a final condition and that we seek to solve this PDE backwards in time. The plus sign in front of the second-order term in (1) will thus not cause any stability problems. A precise mathematical formulation of the problem will be presented below.

For a majority of multi-asset option models the payoff function at expiry is

$$P(S_1, S_2, \ldots, P_n, T) = \phi(S_1, \ldots, S_n) = \max\left(E - \sum_{i=1}^{n}\alpha_i S_i, 0\right), \tag{2}$$