Available online at www.sciencedirect.com
ScienceDirect
Journal of Computational and Applied Mathematics 213 (2008) 1 – 13
JOURNAL OF COMPUTATIONAL AND APPLIED MATHEMATICS
www.elsevier.com/locate/cam

A qualocation method for Burgers' equation

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Received 29 August 2005; received in revised form 8 December 2006

Abstract

In this paper, a qualocation method for the one-dimensional Burgers' equation is proposed. A semidiscrete scheme along with optimal error estimates is discussed. Results of a numerical experiment performed support the theoretical results.

MSC: 65M70; 65M15

Keywords: Qualocation method; Burgers' equation; Optimal error estimates; Gauss quadrature rule; Order of convergence

1. Introduction

In this paper, we consider a qualocation method for Burgers' equation given by

$$u_t - \nu u_{xx} + u u_x = 0, \quad x \in I = (0, 1), \; t \in (0, T], \tag{1.1}$$

with the initial and boundary conditions

$$u(x, 0) = u_0(x), \quad x \in (0, 1), \tag{1.2}$$

$$u(0, t) = u(1, t) = 0, \quad t > 0, \tag{1.3}$$

where the positive number $\nu = 1/Re$ is the coefficient of kinematic viscosity, Re denotes the Reynolds number and u_0 is a given function. Burgers' equation is a one-dimensional version of the Navier–Stokes equation. It is widely used as a simplified model for turbulence, boundary layer behaviour, shock wave formation, convection dominated diffusion phenomena, acoustic attenuation in fog and continuum traffic simulation.

Historically, Burgers' equation was first introduced in [3] who gave its steady state solution. It was then discussed in [6,7] after whom the equation was named, as a simplified model for turbulence. This equation was solved analytically for restricted values of initial conditions independently in [14,10]. Benton and Platzman [4] surveyed the exact solution of the one-dimensional Burgers' equation.

For existence of a unique global solution, we refer to Smoller [22, p. 427, Theorem 21.1]. Moreover it is shown that the solution to (1.1) tends to zero uniformly in [0, 1] as $t \to \infty$ [22]. Higher regularity results for (1.1)–(1.3) can be derived by modifying the analysis given in [17, pp. 123–127] and using appropriate compatibility conditions.

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0377-0427/$ - see front matter
doi:10.1016/j.cam.2006.12.028

Several numerical methods like finite difference methods [13], finite element methods [8], mixed finite element technique [9], Chebyshev spectral collocation methods in [5] and collocation procedures using cubic B-splines [2] are used to derive approximate solution to Burgers' equations. Bressan and Quarteroni have also discussed optimal error estimates in weighted L^2-norm. In this paper, an attempt has been made to apply qualocation method to (1.1)–(1.3), which was introduced in [20] in 1988 for boundary integral equations on smooth curves. Subsequently, Sloan et al. [21] in 1993 extended the method to linear second order two-point boundary value problems, and derived optimal error estimates in $W^{j,p}$, $j = 0, 1, 2$, $1 \leqslant p \leqslant \infty$ norms without the quasi-uniform assumption on the finite element mesh.

A qualocation method is precisely a quadrature based modification of the collocation approximations. This method can also be thought of as a discrete Petrov–Galerkin method using a cubic spline trial space and a piecewise linear test space. Complete discretization is achieved by approximating the integrals by composite two-point Gauss quadrature rule. One practical advantage of this method over the orthogonal cubic spline collocation method [18] is that for a given partition there are only half the number of unknowns and, therefore, it reduces the size of the matrix and hence, the computational cost. Jones and Pani [15] discussed the qualocation method for a second order semilinear two-point boundary value problem. Subsequently, Pani [19] expanded the scope of this method by extending the analysis to parabolic initial and boundary value problems in one space dimension. Recently, a qualocation method is also applied to uni-dimensional single phase Stefan problem by Jones and Pani [16] and optimal error estimates are discussed.

The layout of the paper is as follows. In Section 2, the qualocation method is introduced for Burgers' equation. Optimal error estimates for the semi-discrete scheme are derived in Section 3. Finally, in Section 4, numerical implementation of the scheme is discussed and computational order of convergence is derived.

2. The qualocation method

For our subsequent use, we need the following definitions. Let $W^{m,p}(I)$, $1 \leqslant p \leqslant \infty$, $m \in \mathbb{N}$ denote the standard Sobolev spaces:

$$W^{m,p}(I) = \{\psi \in L^p(I) : D^j\psi \in L^p(I), \;\; j = 1, 2, \ldots, m\}$$

with norm

$$\|\psi\|_{W^{m,p}(I)} = \left(\sum_{j=0}^{m} \|D^j u\|^p_{L^p(I)}\right)^{1/p}, \quad \text{for } 1 \leqslant p < \infty,$$

and for $p = \infty$

$$\|\psi\|_{W^{m,\infty}(I)} = \max_{0 \leqslant j \leqslant m} \|D^j\psi\|_{L^\infty(I)},$$

where $D^j\psi$ denotes the jth derivative of ψ in the sense of distributions.

When $p = 2$, we simply denote $W^{m,2}(I)$ by $H^m(I)$ with the norm $\|\cdot\|_m$. Further,

$$H^1_0(I) = \{\psi \in H^1(I) : \psi(0) = \psi(1) = 0\}.$$

When there is no chance of confusion, we may drop I from the definition of $W^{m,p}(I)$ and call it simply $W^{m,p}$. In the sequel, we shall also use the standard spaces $L^p(0, T; X)$ or simply call $L^p(X)$, where X is a Banach space with norm $\|\cdot\|_X$. The norm on $L^p(X)$ is denoted by $\|\psi\|_{L^p(X)} = (\int_0^T \|\psi\|^p_X)^{1/p}$, for $1 \leqslant p < \infty$ and for $p = \infty$, $\|\psi\|_{L^\infty(X)} = \text{ess sup}_{0 \leqslant t \leqslant T} \|\psi\|_X$.

Let $v \in H^2(I) \cap H^1_0(I)$. Multiplying (1.1) by $-v_{xx}$ and integrating with respect to x over I, we obtain the following formulation: Find $u(t) \in H^1_0(I) \cap H^2(I)$, such that for $t > 0$,

$$\begin{aligned} &-(u_t, v_{xx}) + \nu(u_{xx}, v_{xx}) = (uu_x, v_{xx}) \quad \forall v \in H^2(I) \cap H^1_0(I), \\ &(u(0), v) = (u_0, v) \quad \forall v \in H^2(I) \cap H^1_0(I). \end{aligned} \tag{2.1}$$

Finite dimensional approximation: For $N \geqslant 1$, let

$$\Pi_N = \{0 = x_0 < x_1 < \cdots < x_N = 1\}$$