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Some variants of Cauchy's method with accelerated fourth-order convergence

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Abstract

In this paper, we present some variants of Cauchy's method for solving non-linear equations. Analysis of convergence shows that the methods have fourth-order convergence. Per iteration the new methods cost almost the same as Cauchy's method. Numerical results show that the methods can compete with Cauchy's method.

MSC: 41A25; 65D99

Keywords: Cauchy's method; Newton's method; Non-linear equations; Iterative method

1. Introduction

Solving non-linear equations is one of the most important problems in numerical analysis. In this paper, we consider iterative methods to find a simple root of a non-linear equation $f(x) = 0$, where $f : D \subset \mathbb{R} \to \mathbb{R}$ for an open interval D is a scalar function.

In this paper, we derive the iterative methods by Taylor expansion of $f(x)$

$$f(x) \simeq \sum_{k=0}^{m} \frac{f^{(k)}(x_n)}{k!}(x - x_n)^k = P_m(x), \tag{1}$$

where P_m is the Taylor polynomial of degree m whose kth derivatives agree with f at the point x_n, i.e., $P_m^{(k)}(x_n) = f^{(k)}(x_n)$, $k = 0, \ldots, m$. Let the next approximation x_{n+1} be defined as the root of $P_m(x) = 0$ closest to x_n. By solving $P_1(x) = 0$, Newton's method is obtained:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}. \tag{2}$$

This is an important and basic method [13], which converges quadratically.

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In order to derive higher-order methods, we solve $P_2(x)=0$, namely, $f(x_n)+f'(x_n)(x-x_n)+(1/2!)f''(x_n)(x-x_n)^2=0$ and Cauchy's method [14,11] is obtained:

$$x_{n+1}=x_n-\frac{2}{1+\sqrt{1-2L_f(x_n)}}\frac{f(x_n)}{f'(x_n)}, \tag{3}$$

where

$$L_f(x_n)=\frac{f''(x_n)f(x_n)}{f'(x_n)^2}. \tag{4}$$

This method is an iterative process with cubical convergence.

Moreover, by Taylor approximation of $(1-2L_f(x_n))^{1/2}$, it is easy to obtain that

$$\frac{2}{1+\sqrt{1-2L_f(x_n)}}=\frac{1-\sqrt{1-2L_f(x_n)}}{L_f(x_n)}=\sum_{k\geqslant 0}\binom{\frac{1}{2}}{k+1}(-1)^k2^{k+1}L_f(x_n)^k.$$

Thus, the method obtained in [10] is expressed as

$$x_{n+1}=x_n-\left(\sum_{k=0}^{m}\binom{\frac{1}{2}}{k+1}(-1)^k2^{k+1}L_f(x_n)^k\right)\frac{f(x_n)}{f'(x_n)},\quad m\geqslant 1. \tag{5}$$

This method has $(m+2)$th order convergence for approximating square roots. In the case for $m=1$, a famous iterative process of third order, the Euler–Chebyshev method is obtained. For $m=2$, the method with at least third-order method is obtained:

$$x_{n+1}=x_n-\left(1+\frac{1}{2}L_f(x_n)+\frac{1}{2}L_f(x_n)^2\right)\frac{f(x_n)}{f'(x_n)}. \tag{6}$$

The scheme (6) is of fourth-order for quadratic equations [2]. However, except for quadratic equations, Cauchy's method and its variants, Eqs. (5) and (6), only attain third-order of convergence for the general non-linear equations.

On the other hand, Grau and Noguera [7] find a root of the quadratic equation $f(x_n)+f(z_n)+f'(x_n)(x-x_n)+\frac{1}{2}f''(x_n)(x-x_n)^2=0$ and obtain a variant of Cauchy's method with fifth-order convergence

$$x_{n+1}=x_n-\frac{2(f(x_n)+f(z_n))/f'(x_n)}{1+\sqrt{1-2f''(x_n)(f(x_n)+f(z_n))/f'(x_n)^2}}, \tag{7}$$

where

$$z_n=x_n-\frac{2}{1+\sqrt{1-2L_f(x_n)}}\frac{f(x_n)}{f'(x_n)}$$

is the Cauchy's iterate. This method improves the order of convergence and computational efficiency of Cauchy's method with an additional evaluation of the function.

In this paper, by solving $P_3(x)=0$ and the approach to approximate the third derivative with a finite difference between the second derivatives, we obtain some new variants of Cauchy's method for solving non-linear equations. These methods are proved to have at least fourth-order convergence. Moreover, per iteration the new methods require the same evaluations of the function, its first derivative and second derivative as Cauchy's method although the order of convergence is improved. Consequently, the new methods can compete with Cauchy's method, as we show in some examples.

2. The methods

In what follows, we will derive the new methods and firstly define

$$y_n=x_n-\frac{f(x_n)}{f'(x_n)}, \tag{8}$$