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A new upper bound in the second Kershaw's double inequality and its generalizations

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Abstract

In the paper, a new upper bound in the second Kershaw's double inequality involving ratio of gamma functions is established, and, as generalizations of the second Kershaw's double inequality, the divided differences of the psi and polygamma functions are bounded.

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Keywords: Kershaw's double inequality; Ratio of gamma functions; Psi function; Polygamma function; Divided difference; Upper bound; Refinement; Extension; Generalization; Identric mean; Exponential mean; Generalized logarithmic mean; Completely monotonic function; Logarithmically completely monotonic function

1. Introduction

Recall [14,26] that a function f is said to be completely monotonic on an interval I if f has derivatives of all orders on I and

$$(-1)^n f^{(n)}(x) \geqslant 0, \tag{1}$$

for $x \in I$ and $n \geqslant 0$. Recall [1] that if $f^{(k)}(x)$ is completely monotonic on an interval I for some nonnegative integer k, but $f^{(k-1)}(x)$ is not completely monotonic on I, then $f(x)$ is called a completely monotonic function of kth order on an interval I. Recall also [1,17,18,21,22] that a positive function f is called logarithmically completely monotonic on an interval I if f has derivatives of all orders on I and its logarithm $\ln f$ satisfies

$$(-1)^k [\ln f(x)]^{(k)} \geqslant 0, \tag{2}$$

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for all $k \in \mathbb{N}$ on I. It has been presented explicitly by different approaches in [3,17,18,25] that a logarithmically completely monotonic function must be completely monotonic, but not conversely. The famous Bernstein's Theorem in [26, Theorem 12a, p. 160] states that a function f on $[0, \infty)$ is completely monotonic if and only if there exists a bounded and nondecreasing function $\alpha(t)$ such that

$$f(x) = \int_0^\infty e^{-xt} \, d\alpha(t), \tag{3}$$

a Laplace transform of the measure $\alpha(t)$, for $x \in [0, \infty)$.

The generalized logarithmic mean $L_p(a, b)$ of order $p \in \mathbb{R}$ for positive numbers a and b with $a \neq b$ is defined in [4, p. 385] by

$$L_p(a,b) = \begin{cases} \left[\dfrac{b^{p+1} - a^{p+1}}{(p+1)(b-a)}\right]^{1/p}, & p \neq -1, 0, \\ \dfrac{b-a}{\ln b - \ln a}, & p = -1, \\ \dfrac{1}{e}\left(\dfrac{b^b}{a^a}\right)^{1/(b-a)}, & p = 0. \end{cases} \tag{4}$$

Note that

$$L_1(a,b) = \frac{a+b}{2} = A(a,b), \quad L_{-1}(a,b) = L(a,b) \quad \text{and} \quad L_0(a,b) = I(a,b) \tag{5}$$

are called, respectively, the arithmetic mean, the logarithmic mean, and the identric or exponential mean in the literature. Since the generalized logarithmic mean $L_p(a, b)$ is increasing in p for $a \neq b$, see [4, Theorem 3, pp. 386–387], inequalities

$$L(a,b) < I(a,b) < A(a,b) \tag{6}$$

are valid for $a > 0$ and $b > 0$ with $a \neq b$.

It is well known that the classical Euler's gamma function $\Gamma(x)$ is defined for $x > 0$ as

$$\Gamma(x) = \int_0^\infty e^{-t} t^{x-1} \, dt. \tag{7}$$

The logarithmic derivative of $\Gamma(x)$, denoted by

$$\psi(x) = \frac{\Gamma'(x)}{\Gamma(x)}, \tag{8}$$

is called the psi or digamma function, and $\psi^{(i)}(x)$ for $i \in \mathbb{N}$ are known as the polygamma or multigamma functions. These functions play central roles in the theory of special functions and have lots of extensive applications in many branches, for example, statistics, physics, engineering, and other mathematical sciences.

In [8], two double inequalities

$$\exp[(1-s)\psi(n+1)] \geqslant \frac{\Gamma(n+1)}{\Gamma(n+s)} \geqslant n^{1-s} \tag{9}$$

and

$$(n+1)^{1-s} \geqslant \frac{\Gamma(n+1)}{\Gamma(n+s)} \geqslant n^{1-s} \tag{10}$$

on the ratio of two gamma functions were established for $n \in \mathbb{N}$ and $0 \leqslant s \leqslant 1$.