Available online at www.sciencedirect.com
ScienceDirect
JOURNAL OF COMPUTATIONAL AND APPLIED MATHEMATICS
Journal of Computational and Applied Mathematics 220 (2008) 480–489
www.elsevier.com/locate/cam

The cubic semilocal convergence on two variants of Newton's method☆

Quan Zheng*, Rongxia Bai, Zhongli Liu

College of Sciences, North China University of Technology, Beijing 100041, PR China

Received 17 November 2006; received in revised form 12 March 2007

Abstract

In this paper, we discuss two variants of Newton's method without using any second derivative for solving nonlinear equations. By using the majorant function and confirming the majorant sequences, we obtain the cubic semilocal convergence and the error estimation in the Kantorovich-type theorems. The numerical examples are presented to support the usefulness and significance.

Keywords: Nonlinear equation; Newton's method; Cubic semilocal convergence

1. Introduction

It is a fundamental problem in computational mathematics for solving a nonlinear equation:

$$f(x) = 0, \tag{1.1}$$

where $f(x)$ is continuously differentiable and $f'(x) \neq 0$ in a neighborhood of a real root x^*. The well-known Newton's method approximates the root with quadratic convergence as the following (see [4]):

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \quad n = 0, 1, 2, \ldots, \tag{1.2}$$

where x_0 is some initial guess of the root.

In this paper, we consider a variant of Newton's method (see [5]):

$$x_{n+1} = x_n - \frac{f(x_n) + f(x^*_{n+1})}{f'(x_n)}, \quad n = 0, 1, 2, \ldots, \tag{1.3}$$

☆ Supported in part by Beijing Natural Science Foundation (No. 1072009) and National Key Basic Research and Development Program (2002CB312104).
* Corresponding author.
E-mail addresses: zhengq@ncut.edu.cn (Q. Zheng), bairongxia421@163.com (R. Bai), zhongliliu2@163.com (Z. Liu).

0377-0427/$ - see front matter
doi:10.1016/j.cam.2007.09.003

where $x_{n+1}^* = x_n - \frac{f(x_n)}{f'(x_n)}$ is the intermediate result from an iteration of Newton's method (1.2). And we also consider another variant of Newton's method:

$$x_{n+1} = x_n - \frac{f^2(x_n)}{(f(x_n) - f(x_{n+1}^*))f'(x_n)}, \quad n = 0, 1, 2, \ldots, \tag{1.4}$$

which was called Newton–Secant iteration in [5]. Each of them uses one more evaluation of the function to accelerate Newton's iteration. Their cubic convergence and error equations had been obtained.

Scheme (1.3) can be recognized as a two-step method as the following:

$$\begin{cases} y_n = x_n - \dfrac{f(x_n)}{f'(x_n)}, \\ x_{n+1} = y_n - \dfrac{f(y_n)}{f'(x_n)}. \end{cases} \tag{1.5}$$

And scheme (1.4) is rewritten as the following:

$$\begin{cases} y_n = x_n - \dfrac{f(x_n)}{f'(x_n)}, \\ x_{n+1} = y_n - \dfrac{f(x_n)f(y_n)}{(f(x_n) - f(y_n))f'(x_n)}. \end{cases} \tag{1.6}$$

Other methods that use the derivative $f'(x)$ to accelerate the Newton's iteration were discussed in [1,3,6]. And single-step methods that use the second derivative $f''(x)$ to reach cubic convergence were discussed in [2]. They take N and N^2 more operations than that of (1.3) or (1.4), respectively, supposed that $f(x) = 0$ is a system of N nonlinear equations.

Assume that the function $f(x)$ is defined in an open convex set D, $f : D \subset R \to R$. Let the majorant function be

$$h(t) = \frac{K}{2}t^2 - \frac{t}{\beta} + \frac{\eta}{\beta}, \tag{1.7}$$

where K, β and η are positive constants, such that

$$|f'(x_0)^{-1}| \leqslant \beta, \quad |f(x_0)| \leqslant \frac{\eta}{\beta} \quad \text{for an } x_0 \in D, \tag{1.8}$$

and

$$|f'(x) - f'(y)| \leqslant K|x - y|, \quad \forall x, y \in D. \tag{1.9}$$

We have

Lemma 1.1. *If $\alpha = K\beta\eta \leqslant \frac{1}{2}$, then the function $h(t)$ has positive real roots t^* and t^{**}, and*

$$\eta < t^* = \frac{1 - \sqrt{1 - 2\alpha}}{\alpha}\eta \leqslant \frac{1}{K\beta}, \quad t^* \leqslant t^{**} = \frac{1 + \sqrt{1 - 2\alpha}}{\alpha}\eta. \tag{1.10}$$

By using the majorant functions and confirming the majorant sequences for the two-step methods (1.5) and (1.6), we prove their cubic semilocal convergence and obtain the Kantorovich-type theorems too complete the convergence theories for (1.3) and (1.4) in the following sections.

2. Cubic semilocal convergence of (1.3)

By using scheme (1.5) to find the root of (1.7), we have

$$\begin{cases} s_n = t_n - \dfrac{h(t_n)}{h'(t_n)}, \\ t_{n+1} = s_n - \dfrac{h(s_n)}{h'(t_n)}. \end{cases} \tag{2.1}$$