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A parallel algorithm for large systems of Volterra integral equations of Abel type

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Abstract

A significative number of recent applications require numerical solution of large systems of Abel–Volterra integral equations. Here we propose a parallel algorithm to numerically solve a class of these systems, designed for a distributed-memory MIMD architecture. In order to achieve a good efficiency we employ a fully parallel and fast convergent waveform relaxation (WR) method and evaluate the lag term by using FFT techniques. To accelerate the convergence of the WR method and to best exploit the parallel architecture we develop special strategies. The performances of the resulting code, NSWR4, are illustrated on some examples.

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1. Introduction

In this paper we illustrate a parallel algorithm for distributed-memory MIMD architectures, for the solution of a class of systems of Volterra integral equations (VIEs) of Abel type:

$$y(t) = f(t) + \int_0^t \frac{K(t, s, y(s))}{(t-s)^\alpha}\,\mathrm{d}s, \quad t \in [0, T], \tag{1.1}$$

where $y(t)$, $f(t)$, $K(t, s, u) \in \mathbb{R}^d$, $\forall t, s \in [0, T]$ and $\forall u \in \mathbb{R}^d$, $d \geqslant 1$, $0 \leqslant \alpha < 1$.

VIEs of Abel type model many physical and biological problems, like reaction–diffusion problems [11], the behavior of viscoelastic materials in mechanics, superfluidity problems, the propagation of a flame, soft tissues like mitral valves of the aorta in the human heart (for references see [9]).

At present there are a very few high level software to solve a single Abel–VIE. For example the NAG library offers to users only one FORTRAN routine, D05BDF [16], which solves the scalar equation (1.1) in the special case

$$y(t) = f(t) + \frac{1}{\pi}\int_0^t \frac{K(t-s)}{\sqrt{t-s}} g(s, y(s))\,\mathrm{d}s, \quad t \in [0, T], \tag{1.2}$$

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where $y(t)$, $f(t)$, $g(t,u)$, $K(t) \in \mathbb{R}$, $\forall t \in [0,T]$, $\forall u \in \mathbb{R}$. This routine implements fractional BDF methods of order 4, 5 or 6 and computes the solution $y(t)$ on a mesh of equispaced points [13].

On the other hand, to our knowledge, we do not dispose of software tools to solve *systems* of VIEs of type (1.1) when $y, f, K \in \mathbb{R}^d$, $d > 1$, in spite of the number of their applications. Systems of Abel–VIEs arise from the semidiscretization along space of partial differential equations of fractional order, which model for example some anomalous diffusion and subdiffusion processes [15], or arise from the semidiscretization of Volterra–Fredholm integral equations with singular kernels, some of which occur in the modeling of the coding mechanism in the transmission of nervous signals among neurons [10]. In these cases, in order to obtain a better approximation, the number of mesh points for the discretization has to be large, so we come to a large system (i.e., $d >> 1$).

The numerical treatment of systems of VIEs, in the usual approach (see, for example [2]), requires, at each step of the mesh introduced on $[0,T]$, the solution of a dense (nonlinear) system of dimension d, and the evaluation of the numerical lag term. As a consequence, the main problem in numerically solving system (1.1) is the high computational cost. One way to face this problem is to use parallel architectures. With this aim, during last years parallel methods for VIEs have been proposed: some of them realize a parallelism "across the method" (see for example [8] and references quoted in [6]), while other ones, the iterative waveform relaxation (WR) methods [1,3–5], realize a parallelism "across the system", so realize a massive parallelism, especially useful in solving large systems of VIEs.

On these basis, we have developed a parallel software to solve systems of Abel–VIEs of linear convolution type:

$$y(t) = f(t) + \int_0^t \frac{K(t-s)}{(t-s)^\alpha} y(s)\,\mathrm{d}s, \quad t \in [0,T], \tag{1.3}$$

where $y(t)$, $f(t) \in \mathbb{R}^d$, $K(t) \in \mathbb{R}^{d\times d}$, $\forall t \in [0,T]$, $d >> 1$, $\alpha = \frac{1}{2}$.

The main features of our algorithm are:

- the use of a fully parallel and fast convergent WR method: a discrete non-stationary WR (NSWR) method of Richardson type based on a fractional linear method of order 4 [3,4];
- the evaluation of the numerical lag term by the FFT lag-block technique [12,13] implemented in parallel.

In the following we explain the method we implemented in the code (Section 2). In Section 3 we focus our attention on the special and significant strategies we have developed, in order to accelerate the convergence of the NSWR Richardson method, and in order to reduce the computational cost and the amount of data exchanges among processors. Then we outline the special technique we used to evaluate the lag term at a reduced computational cost and show how we organized this computation in parallel. Section 4 illustrates the organization of the code NSWR4, which implements our algorithm. Section 5 is devoted to the illustration of some numerical examples, carried out to test the convergence properties of the NSWR method and the degree of parallelism of our software. Section 6 contains some concluding remarks and some ideas about the future developments of this work.

2. The discrete NSWR method of Richardson type based on a fractional linear method

We focus our attention on the system of Abel–VIEs (1.3). We assume the given real-valued functions $f(t)$ and $K(t)$ to be at least continuous on $[0,T]$. In these hypotheses Eq. (1.3) admits a unique solution $y(t)$ continuous on $[0,T]$ (for reference see [2,7,14]).

The basic idea of WR methods to solve (1.3) is to construct a sequence $\{y^k\}_{k\in\mathbb{N}}$ of solutions of integral problems derived in a suitable way from the given system (1.3). Of course this approach, to be an attractive one, needs to realize fast convergence of the sequence $\{y^k\}_{k\in\mathbb{N}}$ and cheap computation of any single iteration y^k. Both the goals are achieved by the NSWR methods of Richardson type: the first by choosing suitable non-stationary parameters, the second by fully parallelism, as we soon see.

The NSWR Richardson method is generated by the splitting of the kernel K of (1.3):

$$K(t-s) = \mu_k I + (K(t-s) - \mu_k I) \tag{2.1}$$