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Parameter-based Fisher's information of orthogonal polynomials

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Abstract

The Fisher information of the classical orthogonal polynomials with respect to a parameter is introduced, its interest justified and its explicit expression for the Jacobi, Laguerre, Gegenbauer and Grosjean polynomials found.

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1. Introduction

Let $\{\rho_\theta(x) \equiv \rho(x|\theta); x \in \Omega \subset \mathbb{R}\}$ be a family of probability densities parametrized by a parameter $\theta \in \mathbb{R}$. The Fisher information of $\rho_\theta(x)$ with respect to the parameter θ is defined [5,13] as

$$I(\rho_\theta) := \int_\Omega \left[\frac{\partial \ln \rho(x|\theta)}{\partial\theta}\right]^2 \rho(x|\theta)\,\mathrm{d}x = \int_\Omega \frac{\left[\partial\rho(x|\theta)/\partial\theta\right]^2}{\rho(x|\theta)}\,\mathrm{d}x$$
$$= 4\int_\Omega \left\{\frac{\partial[\rho(x|\theta)^{1/2}]}{\partial\theta}\right\}^2 \mathrm{d}x. \quad (1)$$

This quantity refers to information about an unknown parameter in the probability density. It is a measure of the ability to estimate the parameter θ. It gives the minimum error in estimating the parameter of the probability density $\rho_\theta(x)$ [4]. The notion of Fisher information was introduced by Sir R.A. Fisher in estimation theory [13]. Nowadays, it is being used in numerous scientific areas ranging from statistics, information theory [4] to signal analysis [42] and quantum physics

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[14,15,23]. This information-theoretic quantity has, among other characteristics, a number of important properties, beyond the mere nonnegativity, which deserve to be resembled here.

(1) Additivity for independent events. In the case that $\rho(x, y|\theta) = \rho_1(x|\theta) \cdot \rho_2(y|\theta)$, it happens that

$$I[\rho(x, y|\theta)] = I[\rho_1(x|\theta)] \cdot I[\rho_2(y|\theta)].$$

(2) Scaling invariance. The Fisher information is invariant under sufficient transformations $y = t(x)$, so that

$$I[\rho(y|\theta)] = I[\rho(x|\theta)].$$

This property is not only closely related to the Fisher maximum likelihood method but also it is very important for the theory of statistical inference.

(3) Cramer–Rao inequality [41]. It states that the reciprocal of the Fisher information $I(\rho_\theta)$ bounds from below the mean square error of an unbiased estimator f of the parameter θ; i.e.,

$$\sigma^2(f) \geqslant \frac{1}{I(\rho_\theta)},$$

where $\sigma^2(f)$ denotes the variance of f. This inequality, which lies at the heart of statistical estimation theory, shows how much information the distribution provides about a parameter. Moreover, this says that the Fisher information $I(\rho_\theta)$ is a more sensitive indicator of the localization of the probability density than the Shannon entropy power.

(4) Relation to other information-theoretic properties. The Fisher information is related to the Shannon entropy of $\rho(x|\theta)$ via the elegant de Bruijn's identity [4,24,41]

$$\frac{\partial}{\partial\theta} S(\tilde{\rho}_\theta) = \frac{1}{2} I(\tilde{\rho}_\theta),$$

where $\tilde{\rho}_\theta$ denotes the convolution probability density of any probability density $\rho(x|\theta)$ with the normal density with zero mean and variance $\theta > 0$, and $S(\tilde{\rho}_\theta) := -\int_\Omega \tilde{\rho}_\theta(x) \ln \tilde{\rho}_\theta(x)\,\mathrm{d}x$ is the Shannon entropy of $\tilde{\rho}_\theta(x)$. Moreover, the Fisher information $I(\rho_\theta)$ satisfies, under proper regularity conditions, the limiting property [14]

$$I(\rho_\theta) = \lim_{\varepsilon\to 0} \frac{2}{\varepsilon^2} D(\rho_{\theta+\varepsilon} \| \rho_\theta),$$

where the symbol $D(p\|q) := \int_\Omega p(x) \ln(p(x)/q(x))\,\mathrm{d}x$ denotes the relative entropy or Kullback–Leibler divergence of the probability densities $p(x)$ and $q(x)$. Further connections of the Fisher information with other information-theoretic properties are known; see e.g., [15,40,41].

(5) Applications in quantum physics. The classical orthogonal polynomials appear as the radial part of the wavefunctions which characterize the stationary quantum-mechanical states of numerous physical and chemical systems. It is well known, at least for quantum physicists and a large group of applied mathematicians, that the wavefunctions are the physically admissible solutions of the nonrelativistic Schrödinger equation of motion for these systems, which for polar spherical coordinates can often be separated in radial and angular parts. The square of these wavefunctions is a real probability density which, when the system is electrically charged, denotes the experimentally accessible distribution of charge of the system. Often, this density is essentially the Rakhmanov density of orthogonal polynomials as defined by Eq. (5) of this paper. The quantum-mechanical properties of the physical systems completely depend on the spreading of the charge all over the space, that is, on the distribution of the Rakhmanov density all over the corresponding orthogonality support. Furthermore the charge distribution is mostly controlled by the parameter of the involved orthogonal polynomials. The Fisher information is one of the best estimators of this parameter. Up until now the explicit computation of this information-theoretic measure has not been performed.

In addition, the Fisher information $I(\rho_\theta)$ plays a fundamental role in the quantum-mechanical description of physical systems [8,11,14,15,17,19,20,24,25,28–33,40]. It has been shown

(a) to be a measure of both disorder and uncertainty [14,15] as well as a measure of nonclassicality for quantum systems [19,20],