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Numerical pricing of options using high-order compact finite difference schemes

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Abstract

We consider high-order compact (HOC) schemes for quasilinear parabolic partial differential equations to discretise the Black–Scholes PDE for the numerical pricing of European and American options. We show that for the heat equation with smooth initial conditions, the HOC schemes attain clear fourth-order convergence but fail if non-smooth payoff conditions are used. To restore the fourth-order convergence, we use a grid stretching that concentrates grid nodes at the strike price for European options. For an American option, an efficient procedure is also described to compute the option price, Greeks and the optimal exercise curve. Comparisons with a fourth-order non-compact scheme are also done. However, fourth-order convergence is not experienced with this strategy. To improve the convergence rate for American options, we discuss the use of a front-fixing transformation with the HOC scheme. We also show that the HOC scheme with grid stretching along the asset price dimension gives accurate numerical solutions for European options under stochastic volatility.

MSC: 35A35; 35A40; 65N99

Keywords: European options; American options; High-order compact scheme; Grid stretching; Front fixing

1. Introduction

The Crank–Nicolson scheme is a popular technique used in the numerical pricing of financial contracts in a Black–Scholes framework. A recent work by McCartin and Labadie [8] has focused on the use of the Crandall–Douglas scheme for pricing vanilla options. However, it is well known that the kink at the strike price in the payoff function of various options, causes a lower order rate of convergence for high-order schemes. Recently Oosterlee et al. [9] used a grid stretching transformation described in [15] in combination with a fourth-order spatial discretisation based on a five-point stencil and fourth-order backward differencing formula (BDF4) for time discretisation, to obtain a fourth-order accurate solution for European options. The non-compact scheme gives rise to a system which has a pentadiagonal structure and the time evolution is performed over five time levels requiring option values for four initial time steps. This brings some complications since only payoff values are available.

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Our contribution in this paper is the use of high-order compact schemes [6,13] for the pricing of options under the standard Black–Scholes model and Heston's stochastic volatility model. The high-order compact scheme (HOC) is defined over two time levels similar to the Crank–Nicolson scheme and thus it requires only one initialisation step. An additional advantage is that the method described here leads to tridiagonal linear systems, thus allowing the use of fast tridiagonal solvers. Applying a similar grid stretching transformation as used in [9,15] in combination with the high-order compact discretisation leads to an efficient pricing algorithm for the European option. For the American option pricing problem, we use a time dependent grid stretching transformation proposed in [9] but the linear complementarity problem is solved using an efficient procedure developed in [14] for locating the free-boundary value. We show that this gives an optimal exercise curve comparable to [1]; see also [3] while the curve computed by the method described in [9] is less accurate. However, the grid stretching strategy fails to produce a high-order convergence rate. We therefore consider the use of a front-fixing transformation for which the solution on a fixed domain is smooth thus not requiring any grid stretching transformation. We show that the convergence rate strongly depends on an accurate computation of the free boundary. For European options under the stochastic volatility model of Heston [5], we show that numerical solutions having high accuracy can be obtained with coarse grids.

An outline of this paper is as follows. In Section 2, we review the numerical pricing of European and American options in a Black–Scholes setup. In Section 3 we study the use of high-order compact discretisations [6,13] for the heat and Black–Scholes equations. In Section 4, we discuss the applications of the scheme for pricing American options and in Section 5 we extend the HOC scheme to stochastic volatility European call option problems.

2. Options pricing in the Black–Scholes framework

We assume that the stock price $\{S_t, t \in [0, T]\}$ satisfies the stochastic differential equation

$$\mathrm{d}S_t = (r-\delta)S_t\,\mathrm{d}t + \sigma S_t\,\mathrm{d}W_t, \tag{1}$$

where r is the risk-free rate, δ is the continuous dividend, σ is the volatility parameter, T is the maturity date and $\{W_t\}_{0\leqslant t\leqslant T}$ is a standard Brownian motion. Under the assumptions of a frictionless market, the value of a European option at time t is given by

$$V(S_t, t) = \mathrm{e}^{-(r-\delta)(T-t)}\mathbb{E}^{\mathbb{Q}}[V(S_T, T) \mid \mathscr{I}_t], \tag{2}$$

where $V(S_T, T)$ is the payoff function, $\mathscr{I}_t$ is the filtration generated by the stock price process, and $\mathbb{Q}$ is the equivalent martingale measure. From the Feynman-Kac Theorem [15], it follows that (2) is equivalent to the Black–Scholes PDE

$$\frac{\partial V}{\partial \tau} = \mathscr{L}V, \tag{3}$$

with initial and boundary conditions

$$V(S, 0) = \max(S-E, 0),$$
$$V(0, \tau) = 0,$$

and

$$V(S, \tau) = S\mathrm{e}^{-\delta\tau} - E\mathrm{e}^{-r\tau} \quad \text{as } S \to \infty,$$

for an European call option, where $\tau = T - t$ and the spatial operator $\mathscr{L}$ is

$$\mathscr{L} = \frac{1}{2}\sigma^2 S^2 \frac{\partial^2}{\partial S^2} + (r-\delta)S\frac{\partial}{\partial S} - r.$$

In contrast, the American call problem is posed [17] as a linear complementarity problem (LCP) of the form

$$V_\tau \geqslant \mathscr{L}V,$$
$$V(S, 0) = \Phi(S),$$
$$V(S, \tau) \geqslant V(S, 0),$$