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# On the remainder term of Gauss–Radau quadratures for analytic functions

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**Abstract**

For analytic functions the remainder term of Gauss–Radau quadrature formulae can be represented as a contour integral with a complex kernel. We study the kernel on elliptic contours with foci at the points $\pm 1$ and a sum of semi-axes $\varrho > 1$ for the Chebyshev weight function of the second kind. Starting from explicit expressions of the corresponding kernels the location of their maximum modulus on ellipses is determined. The corresponding Gautschi's conjecture from [On the remainder term for analytic functions of Gauss–Lobatto and Gauss–Radau quadratures, Rocky Mountain J. Math. 21 (1991), 209–226] is proved.


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## 1. Introduction

In this paper we prove a conjecture of Gautschi [1] for the Gauss–Radau quadrature formula

$$\int_{-1}^{1} w(t) f(t)\,\mathrm{d}t = \sum_{\nu=1}^{N} \lambda_\nu f(\tau_\nu) + R_N(f)_w, \tag{1.1}$$

with respect to the Chebyshev weight function of the second kind $w(t) = w_2(t) = \sqrt{1-t^2}$ and with a fixed node at $-1$ (or 1).

Let $\Gamma$ be a simple closed curve in the complex plane surrounding the interval $[-1, 1]$ and $\mathscr{D} = \operatorname{int} \Gamma$ its interior. If the integrand $f$ is analytic in $\mathscr{D}$ and continuous on $\overline{\mathscr{D}}$, then the remainder term $R_N(f)_w$ in (1.1) admits the contour integral

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representation

$$R_N(f)_w = \frac{1}{2\pi \mathrm{i}} \oint_\Gamma K_N(z; w) f(z)\, \mathrm{d}z, \tag{1.2}$$

where the kernel is given by

$$K_N(z; w) = \frac{1}{\omega_N(z; w)} \int_{-1}^{1} \frac{w(t)\omega_N(t; w)}{z - t}\, \mathrm{d}t, \quad z \notin [-1, 1],$$

and $\omega_N(z; w) = \prod_{\nu=1}^{N} (z - \tau_\nu)$. It is clear that

$$K_N(\bar{z}; w) = \overline{K_N(z; w)}. \tag{1.3}$$

The integral representation (1.2) leads to the error estimate

$$|R_N(f)_w| \leqslant \frac{\ell(\Gamma)}{2\pi} \left( \max_{z\in\Gamma} |K_N(z; w)| \right) \left( \max_{z\in\Gamma} |f(z)| \right), \tag{1.4}$$

where $\ell(\Gamma)$ is the length of the contour $\Gamma$. In order to get estimate (1.4), one has to study the magnitude of $|K_N(z; w)|$ on $\Gamma$. Note that the previous formulae hold for every interpolatory quadrature rule with mutually different nodes on $[-1, 1]$.

Many authors have used (1.4) to derive bounds of $|R_N(f)_w|$. Two choices of the contour $\Gamma$ have been widely used: (1) a circle $C_r$ with a center at the origin and a radius $r\ (> 1)$, i.e., $C_r = \{z : |z| = r\}$, $r > 1$, and (2) an ellipse $\mathscr{E}_\varrho$ with foci at the points $\pm 1$ and a sum of semi-axes $\varrho > 1$,

$$\mathscr{E}_\varrho = \{z \in \mathbb{C} | z = \tfrac{1}{2}(\varrho \mathrm{e}^{\mathrm{i}\theta} + \varrho^{-1}\mathrm{e}^{-\mathrm{i}\theta}),\ 0 \leqslant \theta < 2\pi\}. \tag{1.5}$$

When $\varrho \to 1$ the ellipse shrinks to the interval $[-1, 1]$, while with increasing $\varrho$ it becomes more and more circle-like. The advantage of the elliptical contours, compared to the circular ones, is that such a choice needs the analyticity of $f$ in a smaller region of the complex plane, especially when $\varrho$ is near 1.

Since the ellipse $\mathscr{E}_\varrho$ has length $\ell(\mathscr{E}_\varrho) = 4\varepsilon^{-1}E(\varepsilon)$, where $\varepsilon$ is the eccentricity of $\mathscr{E}_\varrho$, i.e., $\varepsilon = 2/(\varrho + \varrho^{-1})$, and

$$E(\varepsilon) = \int_0^{\pi/2} \sqrt{1 - \varepsilon^2 \sin^2\theta}\, \mathrm{d}\theta$$

is the complete elliptic integral of the second kind, estimate (1.4) reduces to

$$|R_N(f)_w| \leqslant \frac{2E(\varepsilon)}{\pi\varepsilon} \left( \max_{z\in\mathscr{E}_\varrho} |K_N(z; w)| \right) \|f\|_\varrho, \quad \varepsilon = \frac{2}{\varrho + \varrho^{-1}}, \tag{1.6}$$

where $\|f\|_\varrho = \max_{z\in\mathscr{E}_\varrho} |f(z)|$. As we can see, the bound on the right-hand side in (1.6) is a function of $\varrho$, so it can be optimized with respect to $\varrho > 1$.

This approach was discussed first for Gaussian quadrature rules, in particular with respect to the Chebyshev weight functions (cf. [4,5])

$$w_1(t) = \frac{1}{\sqrt{1 - t^2}}, \quad w_2(t) = \sqrt{1 - t^2}, \quad w_3(t) = \sqrt{\frac{1 + t}{1 - t}}, \quad w_4(t) = \sqrt{\frac{1 - t}{1 + t}},$$

and later has been extended to Bernstein–Szegő weight functions [8] and some symmetric weight functions including especially the Gegenbauer weight functions [10], as well as to Gauss–Lobatto and Gauss–Radau (cf. [1–3,6,9]), and to Gauss–Turán (cf. [7]) quadrature rules.

In [1] Gautschi considered Gauss–Radau and Gauss–Lobatto quadrature rules relative to the four Chebyshev weight functions $w_i$, $i = 1, 2, 3, 4$, and derived explicit expressions of the corresponding kernels $K_N(z; w_i)$, $i = 1, 2, 3, 4$, in terms of the variable $u = \varrho \mathrm{e}^{\mathrm{i}\theta}$; they are the key for determining the maximum point of $|K_N(z; w_i)|$, $i = 1, 2, 3, 4$, on $\Gamma = \mathscr{E}_\varrho$ given by (1.5). Note that $z = (u + u^{-1})/2$. For Gauss–Lobatto quadratures it was proved that $|K_N(z; w_1)|$ attains