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Conservative arbitrary order finite difference schemes for shallow-water flows

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Abstract

The classical nonlinear shallow-water model (SWM) of an ideal fluid is considered. For the model, a new method for the construction of mass and total energy conserving finite difference schemes is suggested. In fact, it produces an infinite family of finite difference schemes, which are either linear or nonlinear depending on the choice of certain parameters. The developed schemes can be applied in a variety of domains on the plane and on the sphere. The method essentially involves splitting of the model operator by geometric coordinates and by physical processes, which provides substantial benefits in the computational cost of solution. Besides, in case of the whole sphere it allows applying the same algorithms as in a doubly periodic domain on the plane and constructing finite difference schemes of arbitrary approximation order in space. Results of numerical experiments illustrate the skillfulness of the schemes in describing the shallow-water dynamics.

Keywords: Conservative finite difference schemes; Operator splitting; Shallow-water equations

1. Introduction

Consider the classical shallow-water system in the spherical coordinates (λ, φ) written in the divergent form [17]

$$\frac{\partial U}{\partial t}+\frac{1}{R\cos\varphi}\frac{1}{2}\left[\left(\frac{\partial uU}{\partial\lambda}+u\frac{\partial U}{\partial\lambda}\right)+\left(\frac{\partial vU\cos\varphi}{\partial\varphi}+v\cos\varphi\frac{\partial U}{\partial\varphi}\right)\right]-\left(f+\frac{u}{R}\tan\varphi\right)V=-\frac{gz}{R\cos\varphi}\frac{\partial h}{\partial\lambda}, \tag{1}$$

$$\frac{\partial V}{\partial t}+\frac{1}{R\cos\varphi}\frac{1}{2}\left[\left(\frac{\partial uV}{\partial\lambda}+u\frac{\partial V}{\partial\lambda}\right)+\left(\frac{\partial vV\cos\varphi}{\partial\varphi}+v\cos\varphi\frac{\partial V}{\partial\varphi}\right)\right]+\left(f+\frac{u}{R}\tan\varphi\right)U=-\frac{gz}{R}\frac{\partial h}{\partial\varphi}, \tag{2}$$

$$\frac{\partial H}{\partial t}+\frac{1}{R\cos\varphi}\left[\frac{\partial zU}{\partial\lambda}+\frac{\partial zV\cos\varphi}{\partial\varphi}\right]=0. \tag{3}$$

Here the vector $(u, v)^{\mathrm{T}}$, $u=u(\lambda,\varphi,t)$, $v=v(\lambda,\varphi,t)$, determines the velocity of a fluid, $H=H(\lambda,\varphi,t)$ is the fluid's depth, $f=f(\varphi)$ is the Coriolis acceleration, R is the sphere's radius, $h=h(\lambda,\varphi,t)$ is the free surface height, $z:=\sqrt{H}$,

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$U := zu$, and $V := zv$. Besides, it holds $h = H + h_r$, where $h_r = h_r(\lambda, \varphi)$ is the underlying relief's height. Problem (1)–(3) is being studied on a sphere S, with λ as the longitude (positive eastward) and φ as the latitude (positive northward).

It is known that the shallow-water equations (SWEs) describing the dynamics of a non-dissipative and unforced fluid conserve such integral characteristics as the mass, total (kinetic plus potential) energy, and potential enstrophy [9,22]. However, numerical solution to SWEs requires a discretization of the continuous equations and a reduction of problem (1)–(3) to a system of finite difference schemes. It is crucial that the discrete analogues of all the invariants of motion usually stop being invariant, and so the numerical solution may contain additional approximation errors and stimulate nonlinear instability [22]. If the norm related to the total energy is not conserved in time then the numerical solution may differ considerably from the exact one, especially in case of long-term integration [6]. In order to avoid the numerical instability effect, conservative finite difference schemes have to be employed [23].

In the last 40 years there have been suggested several finite difference schemes that conserve some or other integral characteristics of the SWEs. Nevertheless, many of these methods treat the temporal derivatives in (1)–(3) as continuous functions, and therefore those schemes stop being conservative when these derivatives are discretized in time and an explicit temporal approximation is used [1,3,4,10,12–14,21]. More precisely, for the fully discrete model (i.e., discrete both in time and in space) the cited methods involve an explicit approximation in time, and hence only first-order integral characteristics, such as the mass, can be conserved. Yet, even if one employs the Crank–Nicolson approximation [2,11] to conserve the energy, the cited methods turn out to be hard-to-implement.

In this paper we develop a new method for the numerical simulation of shallow-water flows. The method permits conserving the mass and the total energy for the fully discrete shallow-water systems, as well as can be used both in the Cartesian and spherical geometries [17,19,20]. The method essentially involves splitting of the model operator by geometric coordinates and by physical processes [7,8], which provides substantial benefits in the computational cost of solution. Yet, in case of the whole sphere it allows applying the same algorithms as in a doubly periodic domain on the plane. The latter is achieved due to the joint use of the splitting method and two different coordinate maps that leads to solving one-dimensional problems with periodic conditions in each direction. As a result, the numerical algorithm for the whole sphere does not differ from the corresponding algorithm for a doubly periodic domain on the plane. In particular, finite difference schemes of arbitrary approximation order in space can easily be constructed for the whole sphere in the same way as for a doubly periodic planar domain. Due to specially chosen spatial approximations, each split system conserves the mass and the total energy. In fact, an infinite family of such schemes is suggested, which are either linear or nonlinear depending on the choice of certain parameters.

Because of the rigid restrictions on the size of a journal article we will consider the SWEs only on a sphere. A detailed study of the Cartesian case can be found in [19]. We plan to consider hydrodynamic (initial boundary value) problems in the next work. However, one particular case of such limited area problems, namely SWEs in a periodic channel on a sphere, will be considered in the present paper as well.

2. Crank–Nicolson approximation and operator splitting

The differential operator of the shallow-water model (SWM) as a closed system without sources and sinks of energy is an antisymmetric operator. In order to construct conservative finite difference schemes, we apply coordinate splitting of the model operator and separate the process of sphere rotation. Then, we employ the well-known fact that in case of an evolutional closed system

$$\frac{\partial \overline{\xi}}{\partial t} + A\overline{\xi} = 0 \tag{4}$$

with an antisymmetric operator A [5], the only two-layered scheme of the form

$$\frac{\overline{\xi}^{n+1} - \overline{\xi}^{n}}{\tau} + A(\alpha\overline{\xi}^{n+1} + (1-\alpha)\overline{\xi}^{n}) = 0, \quad 0 \leqslant \alpha \leqslant 1, \tag{5}$$

keeping the solution's norm constant is the Crank–Nicolson approximation ($\alpha = \frac{1}{2}$) [2]. Although the Crank–Nicolson scheme is dispersive, this feature is effectively controlled by taking the timestep sufficiently small [11]. So, defining