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Polynomial approximation of C^M functions by means of boundary values and applications: A survey

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Abstract

We collect classical and more recent results on polynomial approximation of sufficiently regular real functions defined in bounded closed intervals by means of boundary values only. The problem is considered from the point of view of the existence of explicit formulas, interpolation to boundary data, bounds for the remainder and convergence of the polynomial series. Applications to some problems of numerical analysis are pointed out, such as nonlinear equations, numerical differentiation and integration formulas, special associated differential boundary value problems. Some polynomial expansions for smooth enough functions defined in rectangles or in triangles of $\mathbb{R}^2$ as well as in cuboids or in tetrahedrons in $\mathbb{R}^3$ and their applications are also discussed.

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1. The problem

Let $\Omega \subset \mathbb{R}^N$, $N \geqslant 1$, be a bounded closed polygonal domain. We denote by $C^M(\Omega)$ the class of continuous functions with continuous partial derivatives in Ω up to a fixed order M. We denote by $\mathscr{P}_{\boldsymbol{x}}^M$ the space of polynomials in $\boldsymbol{x} = (x_1, \dots, x_N)$ of degree not greater than M. It will be clear from the context when M denotes a positive integer number or $M = (m_1, \dots, m_N)$ with $m_i \in \mathbb{N}$, $i = 1, \dots, N$; in the last case $p \in \mathscr{P}_{\boldsymbol{x}}^M$ means that the degree of p with respect each unknown x_i it is at most m_i, $i = 1, \dots, N$.

We seek for explicit expansions

$$f(\boldsymbol{x}) = P_M[f](\boldsymbol{x}) + R_M[f](\boldsymbol{x}), \quad \boldsymbol{x} = (x_1, \dots, x_N) \in \Omega, \tag{1}$$

where $f(\boldsymbol{x}) \in C^M(\Omega)$ and $P_M[f](\boldsymbol{x}) \in \mathscr{P}_{\boldsymbol{x}}^{K_M}$ is a polynomial which depends only on the values that $f(\boldsymbol{x})$ and some of its successive derivatives assume at the relevant boundary points of Ω (i.e. the vertices of the polygonal domain if $N > 1$); $R_M[f](\boldsymbol{x})$ is the exact remainder of the *boundary-type expansion* (1).

Polynomial expansions like (1) have found in the past decades several applications to the applied sciences (physical, engineering, etc.). For this reason we review classical and more recent univariate boundary-type expansions and

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investigate it in Section 2 in the following directions: interpolation to boundary data, bounds for the remainder and expansions in polynomial series, approximations by piecewise polynomials and splines. Moreover, in Section 3 we consider applications of the reviewed expansions to the numerical computation of the roots of nonlinear equations, to the construction of formulas for numerical differentiation and integration, to the approximation of solutions of special associated differential boundary value problems. Finally, in Section 4 we discuss briefly (merely because of reasons of limited space) some extensions of the expansions reviewed in Section 1 to rectangles or triangles in $\mathbb{R}^2$ as well as to cuboids or tetrahedrons in $\mathbb{R}^3$ and give some suggestions on its possible applications for future works: for example, to the production of new kernels of Sard's type or to the construction of *boundary-type cubature formulas*.

2. The univariate case

2.1. Existence of formulas

Since each interval $[a, b] \subset \mathbb{R}$ can be applied one to one onto the unit interval $[0, 1]$ by means of the linear transformation of the variable $t \to x = t - a/(b - a)$, we can set $\Omega = [0, 1]$ without loss of generality. In this section m always denotes a positive integer number. In the following we review all classical univariate boundary value formulas and some recent ones. All these formulas have the nice and important property of exactness on certain polynomial spaces; more precisely, from a simple analysis of the remainder terms it can be deduced that

$$P_m[f] \equiv f \quad \text{for each } f \in \mathscr{P}_x^{k_m} \tag{2}$$

for some $k_m \in \mathbb{N}$. If in addition there exists at least a $f \in \mathscr{P}_x^{k_m+1}$ for which

$$P_m[f] \neq f \tag{3}$$

then following a standard terminology we say that the algebraic degree of exactness of the operator $P_m[\cdot]$ is exactly k_m.

Two-point Hermite formula: As a particular case of the general Hermite interpolation formula [4,37,40,45,60], the two-point Hermite formula is the oldest univariate boundary-type formula (1878). Its formulation requires to fix two integer numbers $m, p,\ m \geqslant 2,\ 1 \leqslant p \leqslant m - 1$; then

$$f(x) = P_m[f](x) + R_m[f](x), \quad x \in [0, 1], \tag{4}$$

where $P_m[f](x)$ is the polynomial of degree not greater than $m - 1$ defined by

$$P_m[f](x) = \sum_{j=0}^{p-1} S_{m,j}^p(x) f^{(j)}(0) + \sum_{j=0}^{m-p-1} (-1)^j S_{m,j}^{m-p}(1 - x) f^{(j)}(1) \tag{5}$$

and

$$R_m[f](x) = \int_0^1 K_m(x, t) f^{(m)}(t)\, \mathrm{d}t. \tag{6}$$

The polynomials $S_{m,j}^p(x),\ 1 \leqslant p \leqslant m - 1$, can be defined, for example, by (see [4, p. 66] for equivalent definitions)

$$S_{m,j}^p(x) = \frac{x^j}{j!}(1 - x)^{m-p} \sum_{k=0}^{p-1-j} \binom{m - p + k - 1}{k} x^k,$$

and the kernel function $K_m(x, t)$ is (see [4, p. 77] for equivalent definitions)

$$K_m(x, t) = \begin{cases} \displaystyle\sum_{j=0}^{p-1} \frac{(-t)^{m-j-1}}{(m - j - 1)!} S_{m,j}^p(x), & t \leqslant x, \\ \displaystyle -\sum_{j=0}^{m-p-1} \frac{(1 - t)^{m-j-1}}{(m - j - 1)!} (-1)^j S_{m,j}^{m-p}(1 - x), & x \leqslant t. \end{cases}$$

The algebraic degree of exactness of the operator $P_m[\cdot]$ is $m - 1$.