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Edge-preserving wavelet thresholding for image denoising☆

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Abstract

In this paper we consider a general setting for wavelet based image denoising methods. In fact, in both deterministic regularization methods and stochastic maximum a posteriori estimations, the denoised image $\hat{f}$ is obtained by minimizing a functional, which is the sum of a data fidelity term and a regularization term that enforces a roughness penalty on the solution. The latter is usually defined as a sum of potentials, which are functions of a derivative of the image. By considering particular families of dyadic wavelets, we propose the use of new potential functions, which allows us to preserve and restore important image features, such as edges and smooth regions, during the wavelet denoising process. Numerical results are presented, showing the optimal performance of the denoising algorithm obtained.

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Keywords: Image denoising; Dyadic wavelets; Thresholding estimators; Potential functions

1. Introduction

We consider a classical problem of image processing: find an estimate $\hat{f}$ of an unknown function f, called the original image, from its noisy measurement $\bar{f}$, where $\bar{f} = f + e$ and e denotes an additive noise function. We want to improve the image quality by removing noise without sacrificing important image features, such as edges and homogeneous regions.

A number of approaches have been proposed to this aim, including stochastic and variational methods, non-linear diffusion filtering and wavelet techniques. The relations between some of the different approaches have also been investigated. In [3] it is proven that the variational formulation in the Besov space $B_1^1(L_1^1(I))$ leads to the classical Donoho and Johnstone wavelet shrinkage method [7], and in [1] the authors show that hard and soft thresholding wavelet estimators correspond to the lower and upper envelopes of a class of penalized least-squares estimators. The relations between anisotropic diffusion and robust statistics are analyzed in [2], where it is shown that the anisotropic diffusion equation is closely related to the error norm and influence function in the robust estimation framework. In [9] the authors consider the correspondence between wavelet shrinkage and non-linear diffusion and derive new wavelet shrinkage functions from existing diffusivity functions, while in [14,15] the relations between soft wavelet shrinkage and total variation denoising are analyzed and conditions for the equivalence of these methods are studied.

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Motivated by the statistical interpretation of anisotropic diffusion, as well as the correspondence between statistical approaches and thresholding wavelet estimators, the purpose of this work is to develop new wavelet denoising methods that exploit the interplay between non-linear diffusion filtering and variational methods in the wavelet domain.

To this end, we consider a variational approach in a deterministic wavelet setting, consisting in the minimization of a functional that is the sum of two terms. The first one ensures that the estimated $\hat{f}$ is a faithful approximation of the original image, and the second represents an a priori constraint which enforces a roughness penalty on the estimate. In order to obtain edge-preserving denoising, the latter is defined as a sum of potentials, which are functions of a derivative of the image. Differently from [1], we make a particular choice of the wavelet transform, which is particularly well suited to maintaining important image features. In fact, we use the two-dimensional (2D) discrete dyadic transform introduced in [8], in which the two components of the wavelet transform of a function $f(x, y)$, at each scale, are proportional to the two components of the gradient of f smoothed at that scale.

This allows us to use for the penalty term the well-known potential functions of the non-linear diffusion filtering methods, hence obtaining new non-linear wavelet estimators that inherit the edge recovery properties of the chosen potential.

In spite of the non-linearity of the method, the minimization of the chosen functional can be realized quite efficiently. In fact, in the wavelet domain the original minimization problem uncouples into a family of 2D independent optimization problems. For the particular choice of the penalty function, these problems are non-quadratic but they can easily be solved by using a simple iterative approach. Making use of the results of [4], it is shown that the proposed iterative procedure turns out to be a simplified version of the half-quadratic regularization method known in the literature, from which it inherits all the convergence properties. The resulting denoising algorithm, consisting in a two-step iterative shrinkage procedure for each couple of dyadic wavelet coefficients, has been applied for the denoising of several test images corrupted by Gaussian noise. The results obtained are very encouraging as they are, both in terms of an objective estimate and from the point of view of the visual quality , among the best results in the existing literature.

The paper is organized as follows: in Sections 2 and 3 we briefly review the main ideas of non-linear diffusion filtering, and we define the discrete dyadic wavelet transform. The variational approach is presented in Section 4, while the characterization of the minimizer and the numerical algorithm are given in Section 5. Section 6 presents a representative set of results obtained for some classical test images.

2. Non-linear diffusion filtering

Diffusion algorithms remove noise from an image $\bar{f}$ by modifying the image via a partial differential equation. In the non-linear case, the method obtains a family $u(x, y, t)$ of filtered versions of $\bar{f}$ as the solution of a non-linear diffusion equation

$$\partial_t = \operatorname{div}[g(|\nabla u|)\nabla u], \quad u(x, y, 0) = \bar{f}(x, y),$$

where $|\nabla u|$ is the gradient magnitude and $g(|\nabla u|)$ is a diffusivity function, whose task is to reduce smoothing as the gradient becomes ever larger, as occurs near the object edges [16]. This allows for the preservation of their contrast and location. Typically g is a non-negative, non-increasing function of the gradient magnitude and its properties characterize the behavior of the corresponding non-linear filter.

Well known diffusivity functions are, for example,

$$g_{\mathrm{Ch}}(|\nabla u|) = \frac{\mu}{\sqrt{\mu^2 + |\nabla u|^2}}, \quad g_{\mathrm{PM}_1}(|\nabla u|) = \frac{\mu^2}{\mu^2 + |\nabla u|^2},$$

$$g_{\mathrm{PM}_2}(|\nabla u|) = \mu^2(1 - \mathrm{e}^{-c|\nabla u|^2/\mu^2})$$

due to Charbonnier [4] and Perona–Malik [12]. All these functions are continuously differentiable, bounded from above by 1, positive, monotonously decreasing and approaching zero for $|x| \to \infty$, but they produce different filtering results.

This is due to the different behavior of the corresponding potential functions [16]. In fact, for the Charbonnier diffusivity, the corresponding potential $\psi(x)$ is convex, while the Perona–Malik diffusivities correspond to potential functions $\psi(x)$ that are only convex for $|x| \leqslant \mu$. (See Fig. 1 for the 1D case.)

This means that the Perona–Malik model may sharpen edges, if their gradient is larger than the *contrast parameter* μ, while the Charbonnier model at most preserves edges by reducing diffusion in correspondence with large gradients.