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On some series representations of the Hurwitz zeta function

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Abstract

A variety of infinite series representations for the Hurwitz zeta function are obtained. Particular cases recover known results, while others are new. Specialization of the series representations apply to the Riemann zeta function, leading to additional results. The method is briefly extended to the Lerch zeta function. Most of the series representations exhibit fast convergence, making them attractive for the computation of special functions and fundamental constants.

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1. Introduction

The Hurwitz zeta function, defined by $\zeta(s,a)=\sum_{n=0}^{\infty}(n+a)^{-s}$ for Re $s>1$ and Re $a>0$, extends to a meromorphic function in the entire complex s-plane. This analytic continuation to C has a simple pole of residue one. This is reflected in the Laurent expansion

$$\zeta(s,a)=\frac{1}{s-1}+\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\gamma_n(a)(s-1)^n, \tag{1}$$

wherein $\gamma_k(a)$ are designated the Stieltjes constants [3,4,9,13,18,20] and $\gamma_0(a)=-\psi(a)$, where $\psi=\Gamma'/\Gamma$ is the digamma function. In certain special cases including $a\geqslant 1$ a positive integer and $a=\frac{1}{2}$ the Hurwitz zeta function may be expressed in terms of the Riemann zeta function $\zeta(s)$, a central subject of analytic number theory.

In this paper we present several series representations of the Hurwitz zeta function. As a by-product, we obtain certain integral results in terms of infinite series. We also find application to the computation of various other special functions and fundamental constants [1,2,6]. Earlier series representations of the Riemann and Hurwitz zeta functions have been obtained by several authors including Landau [12], Ramaswami [14], and Wilton [19], and one of our cases agrees with one of these instances. For a collection of formulae involving series of zeta function values we mention Ref. [16].

We present a family of results that is based upon the same technique.

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Proposition 1. *We have for each indicated domain of a*

(i)

$$\zeta(s,a)=\frac{(a-1)^{1-s}}{s-1}-\frac{1}{\Gamma(s)}\sum_{k=1}^{\infty}\frac{\Gamma(s+k)}{(k+1)!}\zeta(s+k,a),\quad \operatorname{Re} a>1, \tag{2}$$

(ii)

$$\zeta(s,a)=\frac{(a-1/2)^{1-s}}{s-1}-\frac{1}{\Gamma(s)}\sum_{k=1}^{\infty}\frac{\Gamma(s+2k)}{4^k(2k+1)!}\zeta(s+2k,a),\quad \operatorname{Re} a>\frac{1}{2}, \tag{3}$$

(iii)

$$\begin{aligned}\zeta(s,a)&=\frac{1}{w^2(s-1)}\zeta(s-1,a-1/w^4)-\frac{1}{\Gamma(s)}\sum_{k=1}^{\infty}\frac{\Gamma(s+3k)}{9^k(3k+1)!}\zeta(s+3k,a)\\&+\frac{w}{\Gamma(s)}\sum_{k=0}^{\infty}\frac{(-1)^k\Gamma(s+2k)}{(2k+1)!(2w)^{2k+1}}\zeta(s+2k,a+1/2w^4)\\&-\frac{1}{w^2\Gamma(s)}\sum_{k=0}^{\infty}\frac{(-1)^k\Gamma(s+2k-1)}{(2k)!(2w)^{2k}}\zeta(s+2k-1,a+1/2w^4),\quad \operatorname{Re} a>-1/2w^4,\end{aligned} \tag{4}$$

where $w\equiv 3^{1/6}$, *and*

(iv)

$$\begin{aligned}\zeta(s,a)&=2^{s-2}\frac{(2a-1)^{1-s}}{s-1}+\frac{1}{\Gamma(s)}\left[\frac{1}{2}\sum_{k=0}^{\infty}\frac{(-1)^k\Gamma(s+2k)}{4^k(2k+1)!}\zeta(s+2k,a)\right.\\&\left.-\sum_{k=1}^{\infty}\frac{\Gamma(s+4k)}{16^k(4k+1)!}\zeta(s+4k,a)\right],\quad \operatorname{Re} a>\frac{1}{2}.\end{aligned} \tag{5}$$

Each of the series representations in Proposition 1 incorporates the pole of $\zeta(s,a)$ at $s=1$ and is initially valid for at least $\operatorname{Re} s>2$. By analytic continuation they hold for all complex $s\neq 1$.

After the proof of this Proposition we discuss various special cases, corollaries, and extensions.

2. Proof of Proposition 1

We rely on the integral representation for $\operatorname{Re} s+pk>1$ and $\operatorname{Re} a>0$

$$\Gamma(s+pk)\zeta(s+pk,a)=\int_0^{\infty}\frac{t^{s+pk-1}e^{-(a-1)t}}{e^t-1}\,dt. \tag{6}$$

We shortly specialize to $k\geqslant 0$ an integer and $p\geqslant 1$ an integer. We also make use of the equivalent expression [7, p. 362]

$$\int_0^{\infty}x^{\mu-1}e^{-\beta x}(\coth x-1)\,dx=\int_0^{\infty}\frac{x^{\mu-1}e^{-(\beta+1)x}}{\sinh x}\,dx=2^{1-\mu}\Gamma(\mu)\zeta(\mu,\beta/2+1),\quad \operatorname{Re}\beta>0,\quad \operatorname{Re}\mu>1. \tag{7}$$

We consider sums of the form

$$\sum_{k=1}^{\infty}\frac{\Gamma(s+pk)\zeta(s+pk,a)}{(pk+1)!p^{2k}}=\int_0^{\infty}\frac{t^{s-1}e^{-(a-1)t}}{e^t-1}\sum_{k=1}^{\infty}\frac{t^{pk}}{(pk+1)!p^{2k}}\,dt. \tag{8}$$