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On the zeros of complex Van Vleck polynomials

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ABSTRACT

Polynomial solutions to the generalized Lamé equation, *the Stieltjes polynomials*, and the associated *Van Vleck polynomials*, have been studied extensively in the case of real number parameters. In the complex case, relatively little is known. Numerical investigations of the location of the zeros of the Stieltjes and Van Vleck polynomials in special cases reveal intriguing patterns in the complex case, suggestive of a deeper structure. In this article we report on these investigations, with the main result being a proof of a theorem confirming that the zeros of the Van Vleck polynomials lie on special line segments in the case of the complex generalized Lamé equation having three free parameters. Furthermore, as a result of this proposition, we are able to obtain in this case a strengthening of a classical result of Heine on the number of possible Van Vleck polynomials associated with a given Stieltjes polynomial.

1. Introduction

Let $\alpha_1, \ldots, \alpha_n$ be distinct complex numbers, and let $\rho_1, \ldots, \rho_n$ be positive numbers. The *generalized Lamé equation* is the second-order ODE given by

$$\prod_{j=1}^{n}(z-\alpha_j)\phi''(z) + 2\sum_{j=1}^{n}\rho_j\prod_{i\neq j}(z-\alpha_i)\phi'(z) = V(z)\phi(z). \tag{1}$$

According to a result in [10], there exist at most $\frac{(n+k-2)!}{(n-2)!k!}$ polynomials V of degree $n-2$ for which (1) has a polynomial solution ϕ of degree k. These polynomial solutions are often called *Stieltjes* or *Heine–Stieltjes polynomials*, and the corresponding polynomials V are known as *Van Vleck polynomials*.

On the basis of Stieltjes' work, the zeros of the Stieltjes polynomials can be nicely interpreted in terms of the equilibrium positions of an electrostatic system with logarithmic potential [8,18,7,4]. Consider the field generated by n charges $\rho_1, \ldots, \rho_n$ fixed at the positions $\alpha_1, \ldots, \alpha_n$ in $\mathbb{C}$, and k positive unit charges allowed to move freely in $\mathbb{C}$, where the charges repel each other according to the law of logarithmic potential. This means that the charges are not point charges, but are distributed along infinite straight wires perpendicular to the plane $\mathbb{C}$. Then the electrostatic potential of the system is given by

$$W(z_1, \ldots, z_k) := -\log\left[\prod_{j=1}^{n}\prod_{i=1}^{k}|z_i-\alpha_j|^{\rho_j}\prod_{l\neq i}|z_l-z_i|\right]. \tag{2}$$

The equilibrium positions then consist of the points $(z_1, \ldots, z_k) \in \mathbb{C}^k$ for which $\nabla W(z_1, \ldots, z_k) = 0$. By computing explicitly $\nabla W(z_1, \ldots, z_k) = 0$ using the expression (2) for W, we deduce that the equilibrium positions satisfy Niven's equation [19]:

$$\sum_{j=1}^{n}\frac{\rho_j}{z_i-\alpha_j} + \sum_{l\neq i}\frac{1}{z_i-z_l} = 0 \quad (i = 1, \ldots, k). \tag{3}$$

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For an equilibrium position of k charges, $(z_1, \ldots, z_k)$, we may construct the polynomial

$$\phi(z) = \prod_{i=1}^{k}(z - z_i). \tag{4}$$

Recalling a basic fact about polynomials,

$$\phi''(z_i) = 2\phi'(z_i) \sum_{l \neq i} \frac{1}{z_i - z_l},$$

it follows that the polynomials (4) solve the Eq. (1) and are therefore Stieltjes polynomials. In particular, this leads to a one-to-one correspondence between the set of Stieltjes polynomials and the set of equilibrium positions of the electrostatic system described above [8].

To interpret the Van Vleck zeros, notice that if $\phi(z)$ is a Stieltjes polynomial, and ν is a zero of the corresponding Van Vleck polynomial, then

$$\phi''(\nu) + \sum_{j=1}^{n} \frac{2\rho_j}{\nu - \alpha_j}\phi'(\nu) = 0.$$

Thus, either $\nu = \alpha_j$ and $\phi'(\nu) = 0$, or $\phi'(\nu) \neq 0$ and

$$\sum_{j=1}^{n} \frac{2\rho_j}{\nu - \alpha_j} + \sum_{j=1}^{k-1} \frac{1}{\nu - z'_j} = 0,$$

where $z'_1, \ldots, z'_{k-1}$ are the zeros of $\phi'(z)$. We see that if we fix charges $2\rho_1, \ldots, 2\rho_n$ at $\alpha_1, \ldots, \alpha_n$ and unit charges at $z'_1, \ldots z'_{k-1}$, then the Van Vleck zeros represent equilibria of a unit charge allowed to move freely in $\mathbb{C}$. (See [13, cf. Theorem 3.1] for this interpretation of solutions to the above equation.)

In addition to the description of the electrostatic system with logarithmic potential, the Lamé equation (1) arises in other contexts, for example the quantum asymmetric top [1] and the Gaudin spin chains [9], as well as the classical case considered by Lamé in the 1830's of solutions to the Laplace equation on an ellipsoid [19, Chapter XXIII].

For $\alpha_1, \ldots, \alpha_n$ real, Stieltjes [17] showed that the locations of the zeros of the Stieltjes polynomials are completely characterized by their distribution in the subintervals $(\alpha_1, \alpha_2), \ldots, (\alpha_{n-1}, \alpha_n)$. Similar results for the zeros of the Van Vleck polynomials were also obtained in [16]. Obviously, no such result holds when the α_i's are complex numbers, and as one can expect, the situation is more complicated than in the real case. The first result in the complex case was obtained in [15]; he showed that the zeros of any Stieltjes polynomial lie inside the convex hull of the set $\{\alpha_1, \ldots, \alpha_n\}$. Following a similar argument, Marden [12] extended Polya's result to also include the zeros of Van Vleck polynomials. More recently, a refinement of these results for special configurations of the α_i's has also been obtained by Zaheer and Alam [20,21].

In this paper, we are interested in the location of the zeros of the Van Vleck polynomials in the case where the charges ρ_j are located on the vertices α_i of cyclic polygons in the complex plane, where numerical investigations reveal substantial patterns formed by the zeros. In Section 2, we establish some rigorous results on the zero loci of the Van Vleck polynomials when the α_i form the vertices of any equilateral triangle, namely, that the zeros lie on particular portions of the angle bisectors. In Section 3, we discuss non-equilateral triangles, higher order polygons, and the technical difficulties in generalizing the proof of our theorem. In particular, we present some numerical results supporting the conjecture that although there are patterns that the zeros adhere to, the result obtained in Section 2 is unique to case of the equilateral triangle. The final section contains some concluding remarks and conjectures.

2. The equilateral case

Most studies of the generalized Lamé equation (1) consider the case when the charges ρ_i are all positive. We will consider in this section, though, charges that can take on all real values. In the subsection that follows we will focus on positive charges, for which our main result applies. As a simple corollary we derive a remarkable result regarding the case when the fixed charges are zero, namely that the free charges are still distributed in the convex hull of the "zero charges". In the succeeding subsection we consider negative fixed charges.

2.1. Nonnegative charges $\rho_j \geq 0$

Specify a charge strength $\rho > 0$ and 2 locations α_1, α_2 in the complex plane, and place two charges at these locations. Up to reflection across the line passing through α_1 and α_2, a location α_3 for a third charge is determined so that the charges lie at the vertices of an equilateral triangle. Consider further the line segments, henceforth called the *bisectrices*, connecting the vertices to the triangle incenter. Since the Lamé equation is invariant under complex affine transformations, we can assume without loss of generality that the α_j's are the third roots of unity, i.e.

$$\alpha_{j+1} = \mathrm{e}^{\frac{2\pi \mathrm{i}}{3} j} \quad \text{for } j = 0, 1, 2.$$