Journal of Computational and Applied Mathematics 223 (2009) 929–937

Contents lists available at ScienceDirect

Journal of Computational and Applied Mathematics

journal homepage: www.elsevier.com/locate/cam

On the influence of numerical integration on mixed finite element approximations of a Maxwell eigenvalue problem

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ARTICLE INFO

Article history:
Received 25 June 2007
Received in revised form 6 March 2008

MSC:
35P05
65D32
65M60

Keywords:
Mixed finite element approximation
Maxwell eigenvalue problem
Numerical integration

ABSTRACT

The behaviour of electromagnetic resonances in cavities is modelled by a Maxwell eigenvalue problem (EVP). In the present work, we rewrite the corresponding variational problem, as it arises with a view to the application of a finite element method, in a mixed formulation. For the modelling of realistic problems the integrals occurring in this mixed formulation often cannot be evaluated exactly. We take into account the error arising from numerical quadrature and show convergence to the approximations using exact integration. Finally, some numerical results are presented.

1. Introduction

In designing e.g. semiconducting lasers, lossy waveguides and particle accelerators, one encounters electromagnetic resonances in cavities. These resonances are described using the Maxwell equations. In order to reduce development time and costs, a realistic modelling is indispensable.

We will assume the occurring electrical and magnetic fields to be time-harmonic, i.e. the observable fields have the form $\Re(\mathbf{E}e^{i\omega t})$, resp. $\Re(\mathbf{H}e^{i\omega t})$, where $\mathbf{E}$, resp. $\mathbf{H}$, is a complex unknown vector field depending on the space coordinates and ω is the corresponding unknown frequency. Under those assumptions we are led to the following Maxwell EVP:

Problem 1. Find eigentriples $(\omega, \mathbf{E}, \mathbf{H}) \in \mathbb{R} \times \mathbf{L}^2(\Omega) \times \mathbf{L}^2(\Omega)$ such that

$$\nabla \times \mathbf{E} = -i\omega\mu\mathbf{H} \tag{1a}$$

$$\nabla \cdot (\mu\mathbf{H}) = 0 \tag{1b}$$

$$\nabla \times \mathbf{H} = i\omega\epsilon\mathbf{E} \tag{1c}$$

$$\nabla \cdot (\epsilon\mathbf{E}) = 0 \tag{1d}$$

along with the BCs on $\Gamma_\tau \subset \Gamma$

$$\mathbf{E} \times \mathbf{n} = 0 \tag{1e}$$

$$(\mu\mathbf{H}) \cdot \mathbf{n} = 0 \tag{1f}$$

and the BCs on $\Gamma_\nu \subset \Gamma$

$$\mathbf{H} \times \mathbf{n} = 0 \tag{1g}$$

$$(\epsilon\mathbf{E}) \cdot \mathbf{n} = 0. \tag{1h}$$

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doi:10.1016/j.cam.2008.03.021

Here, the material parameters ϵ and μ (dielectricity, resp. magnetic permeability) are bounded, coercive and piecewise Lipschitz continuous tensor fields. The theoretical framework of the paper allows for bounded Lipschitz domains $\Omega \subset \mathbb{R}^3$, but since we will later on use exact triangulations of Ω, for simplicity, we restrict ourselves in reality to polyhedric domains.

It is known, see e.g. [10], that the modelling of electromagnetic resonances is delicate. The early attempts to calculate FEM approximations soon faced the occurrence of non-physical, so-called spurious, eigenmodes. Later research (see [9] and the references therein) demonstrated that the origin of those spurious modes is situated in the infinite dimensionality of the kernel of the curl operator, giving rise to a non-compact resolvent operator.

To overcome these difficulties several methods have been proposed. In general, there are two possibilities: either one imposes the constraint of divergence-freeness on the problem, or one looks for an easy identification of the eigenvectors from the kernel of the curl operator. In order to impose the divergence-free constraint on the problem one may try to incorporate this property in the definition of the discrete function spaces. Several authors, however, prefer to impose this constraint implicitly, using a mixed formulation. In this paper we will follow this last approach.

2. Mixed finite element formulation

In order to state a variational and mixed formulation, we begin with

Definition 1. We consider the following function spaces:

$$\mathbf{H}_{0,\Gamma_\tau}(\text{curl}; \Omega) = \left\{ \mathbf{v} \in \mathbf{L}^2(\Omega) : \nabla \times \mathbf{v} \in \mathbf{L}^2(\Omega), \mathbf{v} \times \mathbf{n}|_{\Gamma_\tau} = 0 \right\}$$
$$\mathbf{H}_{0,\Gamma_\tau}(\text{curl } 0; \Omega) = \{ \mathbf{v} \in \mathbf{H}_{0,\Gamma_\tau}(\text{curl}; \Omega) : \nabla \times \mathbf{v} = 0 \}$$
$$\mathbf{H}_{0,\Gamma_\tau,\Gamma_\nu}(\text{curl; div } 0; \Omega) = \{ \mathbf{v} \in \mathbf{H}_{0,\Gamma_\tau}(\text{curl}; \Omega) : \nabla \cdot \mathbf{v} = 0, (\epsilon \mathbf{v}) \cdot \mathbf{n}|_{\Gamma_\nu} = 0 \}$$
$$W = \left\{ \mathbf{v} \in \mathbf{L}^2(\Omega) : (\mu^{-1/2}\mathbf{v}, \mathbf{u}) = 0, \forall \mathbf{u} \in \mathbf{H}_{0,\Gamma_\tau}(\text{curl } 0; \Omega) \right\}.$$

Furthermore, for simplicity of notation we state

Definition 2. We consider the following inner products, norms and sesquilinear forms:

$$a(\mathbf{u}, \mathbf{v}) = (\epsilon \mathbf{u}, \mathbf{v}), \quad \forall \mathbf{u}, \mathbf{v} \in \mathbf{L}^2(\Omega)$$
$$\gamma(\mathbf{u}, \mathbf{v}) = (\mu^{-1} \nabla \times \mathbf{u}, \nabla \times \mathbf{v}), \quad \forall \mathbf{u}, \mathbf{v} \in \mathbf{H}_{0,\Gamma_\tau}(\text{curl}; \Omega)$$
$$b(\mathbf{u}, \mathbf{v}) = (\mu^{-1/2} \nabla \times \mathbf{u}, \mathbf{v}), \quad \forall \mathbf{u} \in \mathbf{H}_{0,\Gamma_\tau}(\text{curl}; \Omega), \mathbf{v} \in \mathbf{L}^2(\Omega).$$

In [3], one shows that the appropriate variational formulation of Problem 1 is the following one:

Problem 2. Find $(\omega, \mathbf{u}) \in \mathbb{R} \times \mathbf{H}_{0,\Gamma_\tau,\Gamma_\nu}(\text{curl; div } 0; \Omega)$ such that

$$\gamma(\mathbf{u}, \mathbf{v}) = \omega^2 a(\mathbf{u}, \mathbf{v}), \quad \forall \mathbf{v} \in \mathbf{H}_{0,\Gamma_\tau,\Gamma_\nu}(\text{curl; div } 0; \Omega).$$

This is the so-called primal (or electrical) formulation. It is possible to use the dual (or magnetical) formulation using analogous arguments.

The implementation of a finite element method based on Problem 2, however, is far from obvious due to the divergence-free constraint appearing in the space of trial and test functions. The discretization of this constraint has been considered in e.g. [5]. It turns out to be more tractable to consider a strongly related variational problem in which one uses the space $\mathbf{H}_{0,\Gamma_\tau}(\text{curl}; \Omega)$ instead of $\mathbf{H}_{0,\Gamma_\tau,\Gamma_\nu}(\text{curl; div } 0; \Omega)$. This approach has elaborately been considered in [3].

An alternative approach is to include the divergence-free constraint implicitly by passing to a suitable mixed finite element formulation. As has been shown in [2], an equivalent mixed formulation of Problem 2 is the following:

Problem 3. Find $(\omega, \mathbf{u}, \mathbf{p}) \in \mathbb{R} \times \mathbf{H}_{0,\Gamma_\tau}(\text{curl}; \Omega) \times W$ such that:

$$\begin{cases} a(\mathbf{u}, \tau) - b(\tau, \mathbf{p}) = 0 & \forall \tau \in \mathbf{H}_{0,\Gamma_\tau}(\text{curl}; \Omega) \\ b(\mathbf{u}, \mathbf{q}) = \omega^2 (\mathbf{p}, \mathbf{q}) & \forall \mathbf{q} \in W. \end{cases}$$

Moreover, in [2], Problem 3 is proved to be equivalent with Problem 2 in the following sense:

Lemma 1. *If $(\omega, \mathbf{u}, \mathbf{p})$ is a solution of Problem 3, then $(\omega, \mathbf{u})$ is a solution of Problem 2. Conversely, if $(\omega, \mathbf{u})$ is a solution of Problem 2, then $(\omega, \mathbf{u}, \omega^{-2}\mu^{-1/2}\nabla \times \mathbf{u})$ is a solution of Problem 3.*

The convergence of this mixed finite element formulation can be obtained by studying the spectrum of a suitable operator obtained from a related source problem. This operator is introduced as follows: