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A computer-assisted proof for the Kolmogorov flows of incompressible viscous fluid

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ABSTRACT

A computer-assisted proof of non-trivial steady-state solutions for the Kolmogorov flows is described. The method is based on the infinite-dimensional fixed-point theorem using Newton-like operator. This paper also proposes a numerical verification algorithm which generates automatically on a computer a set including the exact non-trivial solution with local uniqueness. All discussed numerical results take into account the effects of rounding errors in the floating point computations.

1. Introduction

Consider the following Navier–Stokes equations:

$$u_t + uu_x + vu_y = \nu\Delta u - \frac{1}{\rho} p_x + \gamma \sin\left(\frac{\pi y}{b}\right), \tag{1}$$

$$v_t + uv_x + vv_y = \nu\Delta v - \frac{1}{\rho} p_y, \tag{2}$$

$$u_x + v_y = 0, \tag{3}$$

where (u, v), ρ, p and ν are velocity vector, mass density, pressure and kinematic viscosity, respectively and γ is a constant representing the strength of the sinusoidal outer force. Also $*_\xi := \partial/\partial\xi (\xi = t, x, y)$ and $\Delta := \partial^2/\partial x^2 + \partial^2/\partial y^2$. The flow region is a rectangle $[-a, a] \times [-b, b]$ and the periodic boundary conditions are imposed in both directions. The aspect ratio is denoted by $\alpha := b/a$.

The above Eqs. (1)–(3) describe the Navier–Stokes flows in a two-dimensional flat torus under a special driving force proposed in [1,7] and have a basic solution which is written as

$$(u, v, p) = (k \sin(\pi y/b), 0, d),$$

where $k := b^2\gamma/(\pi^2\nu)$ and d is any constant. It is known that non-trivial solutions bifurcate from the basic solution at a certain Reynolds number, which is defined below, if and only if $0 < \alpha < 1$ [1]. Okamoto–Shoji [7] computed numerically bifurcation diagrams with the Reynolds number as a bifurcation parameter varying the aspect ratio as a splitting parameter.

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They also strongly suggested stability of the bifurcating solutions for all $0 < \alpha < 1$. Nagatou [3] took a new approach to this stability problem by employing the theory of verified computation and showed that the stability of the bifurcating solutions is mathematical rigorously assured for the cases of $\alpha = 0.4$, 0.7 and 0.8. However, theoretical approach to the non-trivial solutions of the Eqs. (1)–(3) has not been showed up to now.

The aim of this paper is to propose a method to prove the existence and the local uniqueness of the steady-state solutions of the Navier–Stokes Eqs. (1)–(3) for a given Reynolds number and aspect ratio by a computer-assisted proof.

In the previous results [11,6], the author considered Rayleigh–Bénard heat convection model which is known as the Oberbeck–Boussinesq approximations and proposed an approach to prove the existence of the steady-state solutions. In [11, 6], the equation is decomposed into a finite-dimensional part and an infinite-dimensional error part, and if both the parts lead to contraction maps under suitable assumptions, an infinite-dimensional fixed-point theorem implies the existence of the solution in a certain function set. In the self-validating process in computer, Newton-like iteration is executed for the finite-dimensional part, and the computation comes down to solving interval linear systems. However, the method adopted Schauder's fixed-point theorem and the local uniqueness is not assured.

On the other hand, Yamamoto [12] have proposed a method to prove the existence and the local uniqueness of solutions to infinite-dimensional fixed-point equations using computer. However, the algorithm needs a special form of the given finite-dimensional set and it turned out that there is a possibility that the verification algorithm come to an end unsuccessfully even if very fine approximate subspaces are used.

Therefore, this paper will take an alternative verification method using norm estimates in the Newton-like iteration. Note that our verification theorem can be described as a more general form and one may apply it to many kinds of differential equations and integral equations which can be transformed into fixed-point equations. We will discuss them in the forthcoming papers.

We admit that our study in this paper has some restrictions (a driving force, two-dimensional rectangle region, boundary condition, etc.), however, we believe that our idea, not our results themselves, will pave the way to a tool to study the global bifurcation structure for partial differential equations arising in more practical, or even industrial problems.

The contents of this paper are as follows. The Navier–Stokes equations are transformed into a non-dimensional form and the function spaces are defined in Section 2. Constructive a priori error estimates for the linearized problems are described in Section 3, which are needed in numerical computations. A fixed-point formulation and an existence theorem using Newton-like iteration is considered in Section 4. A computable verification condition is given in Section 5. Numerical results which prove the existence of steady-state solutions are described in Section 6. All numerical results discussed take into account the effects of rounding errors in the floating point computations.

2. Non-dimensionalization and function spaces

The letter $\mathbf{T}_\alpha$ denotes the rectangular region $(-\pi/\alpha, \pi/\alpha) \times (-\pi, \pi)$ for a given aspect ratio $0 < \alpha < 1$. Introducing the stream function ϕ satisfying $u = \phi_y$ and $v = -\phi_x$ so that $u_x + v_y = 0$, the Eqs. (1)–(3) can be rewritten as

$$(\Delta\phi)_t - \nu\Delta^2\phi - J(\phi, \Delta\phi) = \frac{\gamma\pi}{b}\cos\left(\frac{\pi y}{b}\right) \tag{4}$$

by cross-differentiating Eqs. (1) and (2) and eliminating the pressure p. Here J is a bilinear form defined by

$$J(u, v) := u_x v_y - u_y v_x. \tag{5}$$

Eq. (4) is non-dimensionalized using change of variables

$$(x', y') = \left(\frac{\pi x}{b}, \frac{\pi y}{b}\right), \qquad t' = \frac{\gamma b}{\nu\pi}t, \qquad \phi'(t', x', y') = \frac{\nu\pi^3}{\gamma b^3}\phi(t, x, y)$$

and the Reynolds number $R := \frac{\gamma b^3}{\nu^2\pi^3}$. After dropping the primes, an equation

$$(\Delta\phi)_t - \frac{1}{R}\Delta^2\phi - J(\phi, \Delta\phi) = \frac{1}{R}\cos(y) \tag{6}$$

is obtained.

We shall find *steady-state solutions*, where $(\Delta\phi)_t$ is equated to 0 in Eq. (6) in the region $\mathbf{T}_\alpha$, namely consider the following non-linear problem:

$$\Delta^2\phi = -RJ(\phi, \Delta\phi) - \cos(y) \quad \text{in } \mathbf{T}_\alpha. \tag{7}$$

Assume that the stream function ϕ is periodic in x and y, and the symmetric condition $\phi(x, y) = \phi(-x, -y)$ [3], then Eq. (7) has a trivial solution $\phi = -\cos(y)$ for any $R > 0$. The aim of this paper is to verify the existence of non-trivial solutions by a computer.