Journal of Computational and Applied Mathematics 223 (2009) 983–996

Contents lists available at ScienceDirect

Journal of Computational and Applied Mathematics

journal homepage: www.elsevier.com/locate/cam

Constrained variational refinement

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A R T I C L E   I N F O

Article history:
Received 15 May 2007
Received in revised form 3 February 2008

MSC:
41A05
41A15
41A29

Keywords:
Splines
Interpolation
Approximation

A B S T R A C T

A non-uniform, variational refinement scheme is presented for computing piecewise linear curves that minimize a certain discrete energy functional subject to convex constraints on the error from interpolation. Optimality conditions are derived for both the fixed and free-knot problems. These conditions are expressed in terms of jumps in certain (discrete) derivatives. A computational algorithm is given that applies to constraints whose boundaries are either piecewise linear or spherical. The results are applied to closed periodic curves, open curves with various boundary conditions, and (approximate) Hermite interpolation.

1. Introduction

In the last few decades, subdivision and other curve refinement schemes have gained prominence, in part due to their connection to wavelets in Approximation Theory, and in part to their applications in areas such as Geometric Design and Computer Graphics. In Approximation Theory, one typically considers real-valued functions, whereas, in Geometric Design, one considers vector-valued functions, i.e. parametric curves and surfaces. In both fields, the subdivision schemes that appear in the literature are most-often uniform (and often stationary). These uniform schemes lead to elegant formulations and analysis in terms of refinement relations and subdivision masks.

In this paper, we consider a non-uniform, variational method for refining curves subject to convex set constraints that is a generalization of the uniform, interpolatory refinement scheme in [11]. We derive optimality conditions, including conditions for optimal free knots, and we use these and simpler methods of parametrization (such as *centripetal parametrizations*) to develop computational algorithms (see [13]). To emphasize the need for non-uniform refinement, one can compare the two curves in Fig. 1.1. Here, the left curve was generated by interpolatory refinement with a uniform parametrization (like in [11]), and the right image using the non-uniform refinement methods described here. Indeed, the need for a non-uniform refinement and subdivision schemes in geometric modeling is akin to the need for non-uniform B-spline curves over uniform splines, for example.

In this paper we generalize 'uniform interpolation' to 'non-uniform near-interpolation'. In particular, we assume that the near-interpolatory constraints are convex. Perhaps the first use of such constraints in spline curve interpolation was in [14,5]. The results in [14] include stationary conditions for general convex sets, for fixed knots, and an algorithm that applies to constraints with piecewise linear boundaries. Similar results are derived in [5], based on a particular construction that leads to fixed and free-knot optimality conditions, and a computational algorithm. It should be noted that since these problems are convex, there are certainly general-purpose optimization algorithms that can be used to compute the curves. The algorithm given here is easy to program, and good for computing approximate solutions.

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doi:10.1016/j.cam.2008.03.033

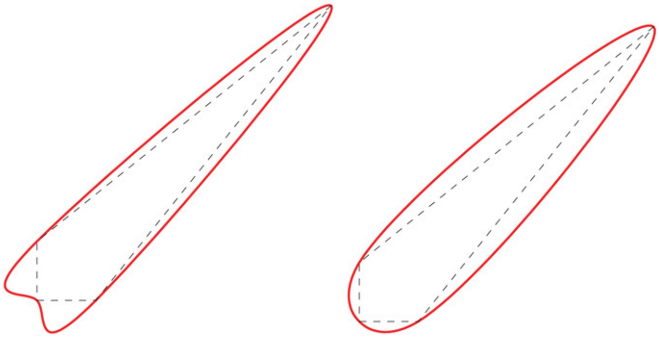

Fig. 1.1. Uniform and non-uniform interpolatory variational refinement.

This paper proceeds as follows: In Section 2, the "energy" of curves is represented in both matrix form, and in terms of certain jumps in third divided differences. An orthogonality condition is derived for the free-knot problem, represented in terms of these jumps. In Section 3, optimality conditions are derived for interpolation, near-interpolation to balls, smoothing, near-interpolation to general convex constraints, and Hermite near-interpolation. Conditions are given for open and closed curves. In Section 4, an algorithm is given for computing approximate solutions. It applies to the case that the constraints are given by balls or convex sets with piecewise linear boundaries. Although we derive optimality conditions for free-knots, these conditions are not easy to implement in computation. Therefore, we prefer to use other methods to update the knots (so-called parameter updates).

A preliminary version of this paper appear in the unpublished manuscript [7]. Optimality conditions for polynomial spline curves under similar constraints as in this paper were given in [5]. For the case of non-uniform interpolatory variational without tension, smoothness conditions are given in [10], and an abstract formulation for constrained variational refinement is given in [8]. Smoothness conditions have not been investigated in the generality described in this paper. Smoothness is also investigated in [9] for a certain class of parametrizations, such as chordal and centripetal parametrizations. The near-interpolatory refinement scheme considered in this paper is generalized to surfaces in [6].

2. Variation in the "energy" of piecewise linear curves

In this section we present the energy functional that we will use to measure the smoothness of the variational refinement curves, and derive optimality conditions for the variation of the functional with respect to both the points of the curves, and the knots. These results are used in the sections that follow.

Now, at each level of refinement we have a sequence of points at certain *knots*. If we connect these points by straight line segments, we have a piecewise linear curve. We assume here for convenience that the curves are closed. Let $f(t)$ be a closed-periodic piecewise linear (B-spline) curve $f(t) = \sum_{i=1}^{n+1} p_i N_{i,1}(t)$ with knots $t_0, \ldots, t_{n+2}$ and coefficients $p_i = f(t_i)$ in $\mathbb{R}^d$. Let $h_i := t_{i+1} - t_i$ and $h_{i,j} := t_{i+j} - t_i$. In particular, $h_{i,1} = h_i$ and $h_{i,2} = t_{i+2} - t_i = h_{i+1} + h_i$. Since f is closed, $p_{n+1} = p_1$, and since it is periodic, $t_{n+2} = t_{n+1} + h_1$ and $t_0 = t_1 - h_n$. Let $\Delta_{i,k} f$ denote the k-th divided difference $\Delta_{i,k} f := [t_i, \ldots, t_{i+k}]f$ of f at knots t_i. In particular, $2\Delta_{i-1,2} f = 2[t_{i-1} t_i, t_{i+1}]f$ is the central difference, centered about t_i. We define the *energy* in the curve as

$$E(f) := \frac{1}{2} \sum_{i=1}^{n} \int_{\frac{t_i+t_{i-1}}{2}}^{\frac{t_i+t_{i+1}}{2}} |2\Delta_{i-1,2} f|^2 \, dt = \sum_{i=1}^{n} |\Delta_{i-1,2} f|^2 h_{i-1,2}$$

This is a discretization for the energy functional $\frac{1}{2}\int_{t_1}^{t_{n+1}} |D^2 f(t)|^2 \, dt$ that is used to characterize best C^2 cubic spline interpolants. The functional $E(f)$ differs from that considered in [12] in the extra term $h_{i-1,2}$ that results from the discretization of the measure in the integral. As it turns out, this term is important in deriving certain conditions in the next section.

The setup is illustrated in Fig. 2.1. The energy of the piecewise linear curves is evaluated by summing over the second-divided differences squared across the dashed lines. On the left, the curve is closed, hence we add an additional point $p_{n+1} = p_1$. On the right, the curve is open, and two of the spans over which the curve is integrated have been removed. Open curves are considered later in Section 3.

We are assuming (for the first part of this paper) that $f(t)$ is closed with periodic knots. To handle higher order divided differences near the end points, we extend the knot sequence to

$$(\ldots, t_{-1}, t_0, \ldots, t_{n+2}, t_{n+3}, \ldots),$$

with the requirement that it wraps periodically, and the coefficient sequence by the requirement $p_{n+k+1} := p_{1+k}$ for $k = 0, \pm 1, \pm 2, \ldots$.