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Pole condition: A numerical method for Helmholtz-type scattering problems with inhomogeneous exterior domain

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Abstract

This paper presents a new numerical method for the solution of exterior Helmholtz scattering problems, which is applicable to inhomogeneous exterior domains and a wide class of geometries. The algorithm is based on the pole condition, which is a general radiation condition and allows a treatment of exterior Helmholtz problems without an explicit knowledge of Green's functions or a series representation. Our algorithm is based on a numerical approximation of the singularities of a Laplace transform of the exterior solution. Numerical examples illustrate the performance of the method.

MSC: 65N35

Keywords: Transparent boundary conditions; Pole condition; Representation formula

1. Introduction

The Helmholtz equation on unbounded domains plays a major role in the mathematical modeling of wave propagation phenomena. It describes an important class of scattering problems in nuclear physics, acoustics and electromagnetic scattering.

Present numerical methods for the solution of these problems are mainly the boundary integral method [1], and the finite element method in combination with infinite elements [2]. An alternative approach is based on the theory of the pole condition by Schmidt [5]. It characterizes outgoing waves by the location of the singularities of the Laplace transform of the exterior field. In [4] it is shown, that the pole condition is equivalent to the Sommerfeld radiation condition for homogeneous exterior domains. For inhomogeneous exterior domains the Sommerfeld radiation condition does not always hold true in contrary to the pole condition approach which covers certain types of inhomogeneous exterior domains. The class of inhomogeneities that can be treated by our approach will be detailed in Section 4.

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We present an algorithmic realization of the pole condition method in a finite element setting. This leads to a system of integro-differential equations for the Laplace transform of the exterior field. The method fits finite element type space discretizations in a natural way and provides a stable back-transformation both in the discrete and continuous setting, which yields a stable representation formula for the exterior field. In its first variant, this method was described in [3]. We generalize this algorithm to the case of nonspherical scatterers and inhomogeneous exterior domains. A coupling to finite elements in an interior domain is possible.

The outline of the paper is as follows. In Section 2 we briefly review basic ideas of the pole condition approach, which originate from [5]. Section 3 describes the finite-element-based part of the method. The underlying concepts were also the basis for a new version of the Perfectly Matched-Layer method [6]. Section 4 details the implementation of the pole condition and Section 5 specifies a new algorithmic realization of the pole condition. Section 6 gives numerical examples which experimentally indicate the convergence of the method.

2. Basic ideas

Consider the exterior Helmholtz equation on an unbounded domain $\Omega = \mathbb{R}^2 \backslash \Omega_{\text{int}}$ with Ω_{int} bounded and a position dependent wave number $k(\boldsymbol{x})$,

$$\begin{aligned} \Delta u(\boldsymbol{x}) + k^2(\boldsymbol{x}) u(\boldsymbol{x}) &= 0 \quad \text{in } \Omega, \\ u(\boldsymbol{x}) &= u_D(\boldsymbol{x}) \quad \text{on } \partial\Omega. \end{aligned} \tag{1}$$

In addition a radiation condition at infinity is required. If $k(\boldsymbol{x}) \equiv k$ is constant for $|\boldsymbol{x}| > R$ for some R a suitable radiation condition is the *Sommerfeld radiation condition*:

$$\lim_{|\boldsymbol{x}|\to\infty} |\boldsymbol{x}|^{1/2}(\partial_\nu u - \mathrm{i}ku) = 0 \text{ uniformly for all directions} \quad \text{where } \nu = \frac{\boldsymbol{x}}{|\boldsymbol{x}|}.$$

In the general setting of a position dependent wave number the radiation condition is provided by the pole condition. The basis of the pole condition is a generalized radial–angular (ξ, η) coordinate system. For a polygonal Ω_{int} the most elementary (ξ, η) coordinate system is obtained from a decomposition of the exterior domain into a finite number of segments Q_j. Each segment Q_j is the image of a reference element under a bilinear mapping $B_j^{\text{loc}}: Q_j^{(\xi,\eta)} \to Q_j^{(x,y)}$ with $Q_j^{(\xi,\eta)} := [0, \infty] \times [\eta_j, \eta_{j+1}]$. The local coordinate transformations have to be constructed such that they can be combined to a globally continuous mapping B, cf. Fig. 1.

We denote the straight nonintersecting rays connecting each vertex $\boldsymbol{p}_j$, $j = 1, \ldots, N_V$ of the polygonal boundary $\partial\Omega_{\text{int}}$ with infinity by $\boldsymbol{g}_j$. Additionally we choose B_j^{loc} such that the images of two parallel lines $\xi_1 \times [\eta_j, \eta_{j+1}] \subset Q_j^{(\xi,\eta)}$ and $\xi_2 \times [\eta_j, \eta_{j+1}] \subset Q_j^{(\xi,\eta)}$ remain parallel under B_j^{loc}. The existence of such a homeomorphism B and

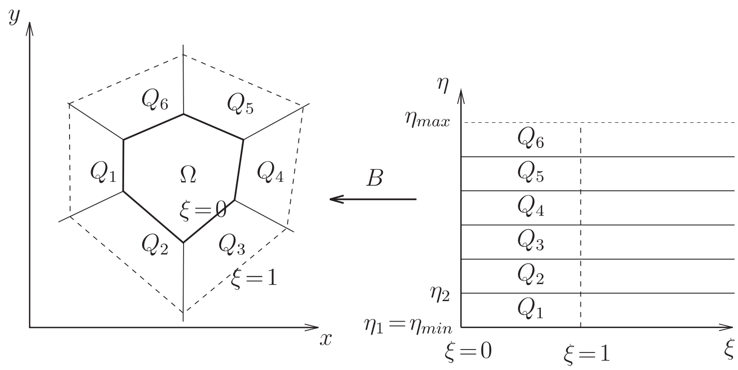

Fig. 1. Prismatoidal coordinate system. Each segment Q_j is the image of a reference element under a bilinear mapping B_j^{loc}. These local mappings are combined to a global mapping B which is continuous in η.