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Regularity properties of the solutions to the 3D Maxwell–Landau–Lifshitz system in weighted Sobolev spaces☆

Ivan Cimrák

NfaM² Research Group for Numerical Functional Analysis and Mathematical Modelling, Department of Mathematical Analysis, Ghent University, Belgium

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Abstract

This paper is concerned with special regularity properties of the solutions to the Maxwell–Landau–Lifshitz (MLL) system describing ferromagnetic medium. Besides the classical results on the boundedness of $\partial_t \mathbf{m}$, $\partial_t \mathbf{E}$ and $\partial_t \mathbf{H}$ in the spaces $L^\infty(I, L^2(\Omega))$ and $L^2(I, W^{1,2}(\Omega))$ we derive also estimates in weighted Sobolev spaces. This kind of estimates can be used to control the Taylor remainder when estimating the error of a numerical scheme.

MSC: 35K55

Keywords: Micromagnetism; Regularity results; Ferromagnets

1. Introduction

We solve full Maxwell–Landau–Lifshitz (MLL) system [9,12–14]

$$\partial_t \mathbf{m} = -\mathbf{m} \times (\Delta \mathbf{m} + \mathbf{H}) - \alpha \mathbf{m} \times (\mathbf{m} \times (\Delta \mathbf{m} + \mathbf{H})), \tag{1}$$
$$\partial_t \mathbf{E} + \sigma \mathbf{E} - \nabla \times \mathbf{H} = \mathbf{0}, \tag{2}$$
$$\partial_t \mathbf{H} + \nabla \times \mathbf{E} = -\beta \partial_t \mathbf{m}, \tag{3}$$
$$\nabla \cdot \mathbf{H} + \beta \nabla \cdot \mathbf{m} = 0, \tag{4}$$
$$\nabla \cdot \mathbf{E} = 0, \tag{5}$$

on the time interval $I = (0, T)$ and inside the domain Ω. Constants α and σ are of the physical origin, representing the dissipation parameter and conductivity of the medium, β is a scaling constant. From physical point of view it is desirable to assume $\alpha > 0$, $\sigma \geqslant 0$. Symbols ∂_t and ∂_t^i denote the first and ith time derivative, respectively. In practical applications σ is a "nice" function describing the conductivity of the medium. In general it varies in space. Nevertheless, a non-constant σ would not change the mathematical analysis and therefore we can consider it as a constant.

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E-mail address: ivan.cimrak@ugent.be.

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We are interested in the cases such as electromagnetic wave propagation, antennas and others, when space-periodic functions occur. Our domain can be then considered as a cube $\Omega = (0, d)^3$ with periodic boundary conditions in each space direction, i.e., for $\mathbf{x} \in \Omega \subset \mathbb{R}^3$, and $t \geqslant 0$,

$$\mathbf{m}(\mathbf{x} + D\mathbf{e}_i, t) = \mathbf{m}(\mathbf{x}, t), \quad \mathbf{H}(\mathbf{x} + D\mathbf{e}_i, t) = \mathbf{H}(\mathbf{x}, t),$$
$$\mathbf{E}(\mathbf{x} + D\mathbf{e}_i, t) = \mathbf{E}(\mathbf{x}, t),$$

where $\mathbf{x} + D\mathbf{e}_i = (x_1, \ldots, x_i + D, \ldots, x_3)$, $i = 1, 2, 3$ and $D > 0$.

The initial conditions read as

$$\mathbf{m}(x, 0) = \mathbf{m}_0(x), \quad \mathbf{H}(x, 0) = \mathbf{H}_0(x), \quad \mathbf{E}(x, 0) = \mathbf{E}_0(x), \quad x \in \Omega \subset \mathbb{R}^3.$$

A crucial observation is, that $|\mathbf{m}| = 1$, for almost all $t \in (0, \infty)$ provided that $|\mathbf{m}_0| = 1$, which is a reasonable physical assumption and that the solution to (1)–(5) is sufficiently smooth. This comes from a scalar multiplication of (1) with $\mathbf{m}$. Then Eq. (1) is equivalent to

$$\partial_t \mathbf{m} - \alpha\Delta\mathbf{m} - \alpha|\nabla\mathbf{m}|^2\mathbf{m} + \mathbf{m} \times \Delta\mathbf{m} = -\mathbf{m} \times \mathbf{H} - \alpha\mathbf{m} \times (\mathbf{m} \times \mathbf{H}). \tag{6}$$

The transformation of Eq. (1) to Eq. (6) is a classical approach used for example in [3,9,18].

1.1. Notations

We use symbols $\langle\cdot,\cdot\rangle$ and $\langle\cdot,\cdot\rangle_i$ for scalar products in $\mathbb{R}^3$ and $\mathbb{R}^i$, respectively. For scalar product in the space $L^2(\Omega)$ we use symbol $(\cdot,\cdot)$. Denote the norms in the spaces $L^p(\Omega)$, $W^{k,p}(\Omega)$ by $\|\cdot\|_p$, $\|\cdot\|_{W^{k,p}}$.

2. Overview of known results

The setting of the problem can vary in BCs. Some authors consider periodic BCs in space, such as Guo and Su. In [11,12] they proved the global existence of a weak solution for the three-dimensional M-LL system. For the case of the Neumann BCs Guo and Ding in [9] are concerned with the global existence and the partial regularity for the weak solution. We mention also strong numerical analysis of Monk and Vacus [17]. They proved the existence of a new class of Liapunov functions for the continuous problem, and then also for a variational formulation of the continuous problem. The authors also showed a special result on continuous dependence.

The case, when the exchange term $\Delta\mathbf{m}$ in the LL equation is considered to be zero, was studied by Joly and others in [13,14]. They show that if the Cauchy data are smooth, then the solution remains smooth for all time. In [15,16] the authors are interested in the numerical modeling of absorbing ferromagnetic materials. They proposed a numerical scheme which conserves the magnitude of magnetization, but they did not prove any error estimates in time. For more details we refer to [22].

In [19,20] the authors suggested a new numerical scheme conserving the magnitude of magnetization and they also proved error estimates in time. They considered the LL equation in a simplified form considering a demagnetizing and anisotropy field but without an exchange field. More results on this scheme can be found also in [6,7,21].

Since the M-LL system is a coupled system of the LL equation and Maxwell's equation, we can expect, that the regularity results of the solutions to the M-LL system can copy those obtained for the single LL equation. We mention work of Visintin [23], Alouges and Soyeur [1], Guo and Hong [10]. All these authors studied the existence of a global weak solution for the LL equation.

For overview of computational micromagnetism also with combination with Maxwell's equations see the monograph [18] written by Prohl.

Carbou and Fabrie in [3] proved the local existence and uniqueness of regular solutions to the LL equation in 3D. So in three dimensions, solutions to the LL equation can blow up in a finite time.

The same results can be obtained also for the M-LL system. We study this case in [4]. For the exact solution of the MLL system several regularity results are known. In the next theorem we recall our results from [4]. We obtained local existence and uniqueness of a weak solution to the system (1)–(5). For strong solutions the same results are valid.

Theorem 1. *Suppose that the initial conditions satisfy*

$$\nabla \cdot \mathbf{E}_0 = \nabla \cdot (\mathbf{H}_0 + \beta\mathbf{m}_0) = 0$$