Available online at www.sciencedirect.com
ScienceDirect
Journal of Computational and Applied Mathematics 216 (2008) 451 – 466
JOURNAL OF COMPUTATIONAL AND APPLIED MATHEMATICS
www.elsevier.com/locate/cam

Some new nonlinear Volterra–Fredholm-type discrete inequalities and their applications

Qing-Hua Ma

Faculty of Information Science and Technology, Guangdong University of Foreign Studies, Guangzhou 510420, PR China

Received 22 February 2006; received in revised form 26 March 2007

Abstract

Some new explicit bounds on solutions to a class of new nonlinear Volterra–Fredholm-type discrete inequalities are established, which can be used as effective tools in the study of certain sum–difference equations. Application examples are also indicated.

MSC: 26D15; 26D20; 39A10

Keywords: Nonlinear discrete inequalities; Volterra–Fredholm-type; Gronwall–Bellman–Ou-Iang-type; Explicit bounds; Sum–difference equations

1. Introduction

In the study of ordinary differential equations, integral equations and difference equations, one often deal with certain integral inequalities. The Gronwall–Bellman inequality [9,2] and its various linear and nonlinear generalizations are crucial in the discussion of the existence, uniqueness, continuation, boundedness, oscillation and stability and other qualitative properties of solutions of differential and integral equations. The literature on such inequalities and their applications is vast; see [1,19,22] and the references therein. A specific branch of this type integral inequalities is originated by Ou-Iang. In his study of boundedness of solutions to linear second order differential equations, Ou-Iang [20] established and used the following nonlinear integral inequality, which is now known as Ou-Iang's inequality in the literature.

Theorem A (*Ou-Iang [20]*). *Let u and f be real-valued, nonnegative, and continuous functions defined on $R_+ = [0, +\infty)$ and let $c \geqslant 0$ be a real constant. Then the nonlinear integral inequality*

$$u^2(t) \leqslant c^2 + 2\int_0^t f(s)u(s)\,\mathrm{d}s, \quad t \in R_+$$

implies

$$u(t) \leqslant c + \int_0^t f(s)\,\mathrm{d}s, \quad t \in R_+.$$

E-mail address: gdqhma@21cn.com.

0377-0427/$ - see front matter
doi:10.1016/j.cam.2007.05.021

While Ou-Iang's inequality is having a neat form and is interesting in its own right as an integral inequality, its importance lies equality heavily on its many beautiful applications in differential and integral equations (see, e.g., [22]). Since this, over the years, many generalizations of Ou-Iang's inequality to various situations have been established; see, for example, [3–7], [11–18], [21–25] and the references cited therein.

Among various generalizations of Ou-Iang's inequality, discrete analogue is also an interesting direction. The point is, similar to the noteworthy contributions of the continuous version of the inequality to the study of differential equations, one naturally expects that discrete versions of the inequality should also play an important role in the study of difference equations. In this respect, fewer results have been established. Recent results in this direction include the works of Pachpatte [21], Pang–Agarwal [23], Ma [13], Meng–Li [17], Cheung [3], Cheung–Ren [7], and Ma–Cheung [14].

The aim of the present paper is to give some explicit bounds to some new nonlinear discrete inequalities involving two-variable functions which, on the one hand, generalize Ou-Iang's inequality to Volterra–Fredholm form at the first time to literatures as we know and, on the other hand, give a handy and effective tool for the study of quantitative properties of solutions of sum–difference equations. We illustrate the usefulness of these inequalities by applying them to study the boundedness, uniqueness, and continuous dependence of the solutions of certain Volterra–Fredholm-type sum–difference equations.

2. Nonlinear discrete inequalities

Throughout this paper, $I := [m_0, M) \cap Z$ and $J := [n_0, N) \cap Z$ are two fixed lattices of integral points in R, where $m_0, n_0 \in Z, M, N \in Z \cup \{\infty\}$. Let $\Omega := I \times J \subset Z^2$, $R_+ := [0, \infty)$, $R_1 := [1, \infty)$ and for any $(s, t) \in \Omega$, the sub-lattice $[m_0, s] \times [n_0, t] \cap \Omega$ of Ω will be denoted as $\Omega_{(s,t)}$.

If U is a lattice in Z (resp.Z^2), the collection of all R-valued functions on U is denoted by $\mathscr{F}(U)$, that of all R_+-valued functions by $\mathscr{F}_+(U)$, and that of all R_1-valued functions by $\mathscr{F}_1(U)$. For the sake of convenience, we extend the domain of definition of each function in $\mathscr{F}(U)$ and $\mathscr{F}_+(U)$ trivially to the ambient space Z (resp.Z^2). So, for example, a function in $\mathscr{F}(U)$ is regarded as a function defined on Z (resp.Z^2) with support in U. As usual, the collection of all continuous functions and all i-times continuously differentiable functions of a topological space X into a topological space Y will be denoted by $C(X, Y)$ and $C^i(X, Y)$, respectively.

If U is a lattice in Z^2 , the partial difference operators Δ_1 and Δ_2 on $u \in \mathscr{F}(Z^2)$ or $\mathscr{F}_+(Z^2)$ are defined as

$$\Delta_1 u(m, n) = u(m + 1, n) - u(m, n), (m, n) \in U,$$

$$\Delta_2 u(m, n) = u(m, n + 1) - u(m, n), (m, n) \in U.$$

For $w \in C(R_+, R_+)$, the function G_1 is defined as

$$G_1(v) = \int_{v_0}^{v} \frac{\mathrm{d}s}{w(s)}, \quad v \geqslant v_0 > 0.$$

Theorem 2.1. *Suppose that u and* $a \in \mathscr{F}_+(\Omega)$, $k \geqslant 0$ *constant and* $w \in C(R_+, R+)$ *is nondecreasing with* $w(r) > 0$ *for* $r > 0$;

$$G_1(\infty) = \int_{v_0}^{\infty} \frac{\mathrm{d}s}{w(s)} = \infty$$

and

$$H_1(t) = G_1(2t - k) - G_1(t) \tag{2.1}$$

is strictly increasing for $t \geqslant k$.