

Limiting properties of the second order Ginzburg–Landau minimizers

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Received 5 July 2006; received in revised form 27 February 2007

Abstract

The author studies the weak convergence for the gradient of the minimizers for a second order energy functional when the parameter tends to 0. And this paper is also concerned with the location of the zeros and the blow-up points of the gradient of the minimizers of this functional. Finally, the strong convergence of the gradient of the radial minimizers is obtained.

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MSC: 35J35; 49J45; 49Q20

Keywords: Second order energy functional; Location of zeros and the blow-up points; Radial minimizer

1. Introduction

Let $G \subset \mathbf{R}^2$ be a bounded and simply connected domain with smooth boundary ∂G . Let g be a smooth map from ∂G into $S^1 = \{x \in \mathbf{R}^2; |x| = 1\}$ and satisfy $\deg(g, \partial G) = d \neq 0$. Without loss of generality, we may assume $d > 0$. Many papers studied the asymptotic behavior of the minimizer m_ε of the lower order Ginzburg–Landau functional

$$E_1(m, G) = \frac{1}{2} \int_G |\nabla m|^2 dx + \frac{1}{4\varepsilon^2} \int_G (1 - |m|^2)^2 dx,$$

in the space $H_g^1(G, \mathbf{R}^2) = \{m \in H^1(G, \mathbf{R}^2); m|_{\partial G} = g\}$. It is easy to get the Euler equations

$$-\Delta m = \frac{1}{\varepsilon^2} m(1 - |m|^2). \quad (1.1)$$

Recall three main results in [4]:

- (R1) The zeros of m_ε are located near d points $a_1, a_2, \dots, a_d \in G$, and $\deg(m_\varepsilon, a_i) = 1$.
- (R2) Although $E_1(m_\varepsilon, G) \rightarrow \infty$ when $\varepsilon \rightarrow 0$, we have the uniform estimation $E_1(m_\varepsilon, K) \leq C$, where K is an arbitrary compact subset of $G \setminus \bigcup_i \{a_i\}$, and $C > 0$ is independent of ε .
- (R3) The limit function u_* of m_ε when $\varepsilon \rightarrow 0$ is a harmonic map.

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If replacing m in the second term of $E_1(m, G)$ by ∇u , as in [10,12], then we have

$$E_2(m, G) = \frac{1}{2} \int_G |\nabla m|^2 dx + \frac{1}{4\varepsilon^2} \int_G (1 - |\nabla u|^2)^2 dx,$$

where $u \in H^2(G, \mathbf{R})$ is determined by the static Maxwell equations, written in the form

$$\Delta u = \operatorname{div} m \quad \text{in } \mathbf{R}^2 \quad (1.2)$$

for the extension of m by 0 outside of G . For a vector field $m \in H^1(G, \mathbf{R}^2)$, we denote the unique weak solution to (1.2) by $u(m)$. Clearly, the minimizer m_ε of $E_2(m, G)$ with $u = u(m)$ in the space $H_g^1(G, \mathbf{R}^2)$ exists. By (1.2) and the method of calculus of variations, it is not difficult to deduce that m_ε satisfies

$$\int_G \nabla m \nabla \zeta dx = \frac{1}{\varepsilon^2} \int_G (1 - |\nabla u(m)|^2) \nabla u(m) \nabla u(\zeta) dx, \quad \forall \zeta \in C_0^\infty(G, \mathbf{R}^2). \quad (1.3)$$

If replacing m in the first term of $E_2(m, G)$ by ∇u , then we have

$$E_\varepsilon(u, G) = \frac{1}{2} \int_G |\nabla^2 u|^2 dx + \frac{1}{4\varepsilon^2} \int_G (1 - |\nabla u|^2)^2 dx.$$

We are concerned with the minimization of the second order functional $E_\varepsilon(u, G)$ in the function class

$$W = \{u \in H^2(G, \mathbf{R}); u|_{\partial G} = g_1, \partial_\nu u|_{\partial G} = g_2\},$$

where $g_1, g_2 \in C^\infty(\partial G, \mathbf{R})$ satisfies $(\partial_\tau g_1, g_2) = g$. By the theories of calculus of variations, the minimizer u_ε on W exists, and it is a weak solution to

$$-(\nabla^2)^2 u = \frac{1}{\varepsilon^2} \operatorname{div}[\nabla u(1 - |\nabla u|^2)]. \quad (1.4)$$

Namely, the minimizer u_ε satisfies the integral equality

$$\int_G \nabla^2 u \nabla^2 \zeta dx = \frac{1}{\varepsilon^2} \int_G \nabla u \nabla \zeta (1 - |\nabla u|^2) dx \quad \forall \zeta \in C_0^\infty(G, \mathbf{R}). \quad (1.5)$$

One of the transformation of the functional $E_\varepsilon(u, G)$ is

$$E_3(u) = \int_G \varepsilon |\nabla^2 u|^2 dx + \int_G \frac{(1 - |\nabla u|^2)^2}{\varepsilon} dx.$$

It is applied to investigate the thin film blisters (cf. [12,13]), where the function u stands for the height of the blistered film and $\int_G |\nabla^2 u|^2 dx$ represents the bending energy. This functional is also used to research the smectic type liquid crystals (cf. [8]). In this case, $m = \nabla u$ where the level sets of u represents the layer structure in the sample. In order to study the energy $\int_G |\nabla^2 u|^2 dx$ under the constraint $|\nabla u| = 1$, it is natural to introduce the penalized energy $E_3(u)$ or $\varepsilon E_3(u)$. The functional $E_3(u)$ is also introduced in the study of the singular perturbation models abstracted from other problems of phase transitions, such as gradient strain theories in plasticity, ferromagnetics, and the areas of materials science and engineering (cf. [1,3,6,7,11]). In those gradient theory of phase transition, the Γ -convergence of the energy was well studied when $\varepsilon \rightarrow 0$. In particular, for certain domains and boundary conditions, the energy $E_3(u)$ concentrates on the defect set of ∇u , and folds of the energy are one-dimensional (cf. [3,11]).

In this paper, we are not concerned with the Γ -convergence of $\varepsilon^2 E_\varepsilon(u, G)$ as dealing with $\varepsilon E_3(u)$, or the fold energy $E_3(u)$ in [11]. We shall discuss the asymptotic properties of ∇u_ε when $\varepsilon \rightarrow 0$, where u_ε is a minimizer of $E_\varepsilon(u, G)$ in W as in [4, Chapter VIII.4]. Namely, we shall consider the special assumption:

$$G = B = \{x \in \mathbf{R}^2; |x| < 1\}, \quad g|_{\partial B} = x, \quad (1.6)$$

which implies $d = 1$. Although the dimension of ε in $E_\varepsilon(u, B)$ is the same as $E_3(u)/\varepsilon$, we do not expect the analysis methods to them are still valid. Indeed, if we notice that the function classes, in particular the boundary conditions equipped to each functional, are different, then the result $E_3(u)/\varepsilon \geq C\varepsilon^{-1}$ (cf. [11]) is not true for the minimal energy

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