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A new lower bound in the second Kershaw's double inequality

Feng Qi[a, b]

[a] College of Mathematics and Information Science, Henan University, Kaifeng City, Henan Province, 475001, China
[b] Research Institute of Mathematical Inequality Theory, Henan Polytechnic University, Jiaozuo City, Henan Province 454010, China

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Abstract

In the paper, a new and elegant lower bound in the second Kershaw's double inequality is established, some alternative simple and polished proofs are given, several deduced functions involving the gamma and psi functions are proved to be decreasingly monotonic and logarithmically completely monotonic, and some remarks and comparisons are stated.

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1. Introduction

In [6], the following double inequalities were established:

$$\left(x+\frac{s}{2}\right)^{1-s}<\frac{\Gamma(x+1)}{\Gamma(x+s)}<\left(x-\frac{1}{2}+\sqrt{s+\frac{1}{4}}\right)^{1-s}, \tag{1}$$

$$\exp[(1-s)\psi(x+\sqrt{s})]<\frac{\Gamma(x+1)}{\Gamma(x+s)}<\exp\left[(1-s)\psi\left(x+\frac{s+1}{2}\right)\right], \tag{2}$$

where $0<s<1$, $x\geqslant 1$, Γ is the classical Euler's gamma function, and ψ is the logarithmic derivative of Γ. They are called the first and second Kershaw's double inequality, respectively. There have been a lot of literature about these two double inequalities and their history, background, refinements, extensions, generalizations and applications. For more detailed information, refer to [9,10,14,15] and the references therein.

E-mail addresses: qifeng@hpu.edu.cn, qifeng618@gmail.com, qifeng618@hotmail.com, qifeng618@msn.com, qifeng618@qq.com.
URL: http://rgmia.vu.edu.au/qi.html.

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The first main result of this paper is the following extension and refinement of the second Kershaw's double inequality (2), which establishes a new and elegant lower bound of inequality (2).

Theorem 1. *For positive numbers s and t with* $s \neq t$,

$$e^{\psi(L(s,t))} < \left[\frac{\Gamma(s)}{\Gamma(t)}\right]^{(s-t)} < e^{\psi(A(s,t))}, \tag{3}$$

where

$$L(s,t) = \frac{s-t}{\ln s - \ln t} \quad \text{and} \quad A(s,t) = \frac{s+t}{2} \tag{4}$$

are, respectively, the logarithmic mean and arithmetic mean of two positive numbers s and t with $s \neq t$. *Equivalently, for* $s, t \in \mathbb{R}$ *and* $x > -\min\{s,t\}$ *with* $s \neq t$,

$$e^{\psi(L(s,t;x))} < \left[\frac{\Gamma(x+s)}{\Gamma(x+t)}\right]^{1/(s-t)} < e^{\psi(A(s,t;x))}, \tag{5}$$

where $L(s,t;x) = L(x+s, x+t)$ *and* $A(s,t;x) = A(x+s, x+t)$ *for* $s, t \in \mathbb{R}$ *and* $x > -\min\{s,t\}$ *with* $s \neq t$.

Recall [12,13,16] that a function f is said to be logarithmically completely monotonic on an interval I if its logarithm $\ln f$ satisfies $(-1)^k[\ln f(x)]^{(k)} \geqslant 0$ for $k \in \mathbb{N}$ on I. It has been proved in [4,11–13] that a logarithmically completely monotonic function on an interval I must be completely monotonic on I. The logarithmically completely monotonic functions have close relationships with both the completely monotonic functions and Stieltjes transforms. For detailed information, refer to [4,11,8,18,21] and the references therein.

The second main result of this paper is to prove the monotonicity of the following two functions, which is a generalization of Theorem 1.

Theorem 2. *For* $s, t \in \mathbb{R}$ *with* $s \neq t$, *the function*

$$\left[\frac{\Gamma(x+s)}{\Gamma(x+t)}\right]^{1/(s-t)} \frac{1}{e^{\psi(L(s,t;x))}} \tag{6}$$

is decreasing and

$$\left[\frac{\Gamma(x+s)}{\Gamma(x+t)}\right]^{1/(t-s)} e^{\psi(A(s,t;x))} \tag{7}$$

is logarithmically completely monotonic in $x > -\min\{s,t\}$.

By the way, a stronger conclusion than [3, Theorem 2.1] is obtained below.

Theorem 3. *Let*

$$f(x) = \frac{\Gamma(x)}{\exp\{[\psi(x)-1]\exp[\psi(x)]\}} \tag{8}$$

for $x \in (0,\infty)$ *and* $c = 1.462632\ldots$ *stand for the unique positive zero of the psi function* ψ. *Then the function* $f(x)$ *is decreasing in* $(0,c)$ *and increasing in* (c,∞) *with*

$$\lim_{x\to 0^+} f(x) = \infty \quad \text{and} \quad \lim_{x\to\infty} = \sqrt{2\pi}. \tag{9}$$

Consequently, for $x \in (0,\infty)$,

$$\Gamma(x) \geqslant \Gamma(c)\exp\{[\psi(x)-1]\exp[\psi(x)]+1\}. \tag{10}$$

In next section, we shall employ simple methods and polished techniques to verify these theorems.