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Algorithms for infinite quadratic programming in L_p spaces

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Abstract

We study infinite dimensional quadratic programming problems of an integral type. The decision variable is taken in the L_p space where $1 < p < \infty$. In this paper the decision variable is required to have a lower bound and an upper bound on a compact interval. Two numerical algorithms are proposed for solving these problems, and the convergence properties of the proposed algorithms are given. Two numerical examples are also given to implement the proposed algorithms.

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1. Introduction

Let X and Y be compact intervals. For $p \geqslant 1$, the space $L_p(X)$ consists of those real-valued measurable functions f on the compact interval X for which $|f(x)|^p$ is a Lebesgue integrable function. The norm on this space is defined as $\|f\|_{L_p} = (\int_X |f(x)|^p \, \mathrm{d}x)^{1/p}$, and we call $\|f\|_{L_p}$ the L_p-norm of f. Now we consider the following infinite dimensional quadratic programming problem. Let $\phi(s, y)$ be a real-valued continuous function on $X \times Y$, $g(y)$ be a real-valued continuous function on Y, $h(s)$ be a real-valued continuous function on X, and $f(s,t)$ be a real-valued continuous function on $X \times X$. Then the infinite dimensional quadratic programming problem (P) is as follows:

$$\min_{k \in L_p(X)} \quad \frac{1}{2} \int_X \int_X f(s,t) k(s) \, \mathrm{d}s k(t) \, \mathrm{d}t + \int_X h(s) k(s) \, \mathrm{d}s$$

$$\text{s.t.} \quad \int_X \phi(s, y) k(s) \, \mathrm{d}s \geqslant g(y) \quad \text{for each } y \in Y,$$

$$0 \leqslant M_1 \leqslant k(s) \leqslant M_2 \text{ a.e.} \quad \text{on } X.$$

Here, M_1 and M_2 are given constants. In this paper, we only consider the case that $1 < p < \infty$. This is an infinite dimensional quadratic programming problem of an integral type. Lai and Wu [6] studied the infinite dimensional linear programming problems on measure spaces, and the necessary and sufficient conditions for a measure to be optimal

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were established in their paper. Meanwhile, solving the general capacity problem by relaxed cutting plane approach can be found in Fang et al. [3]. Ito et al. [5] considered infinite dimensional linear programs in L_1 spaces, while Vanderbei [11] investigated an optimization problem for the best high-contrast apodization. This is an infinite dimensional linear programming problem in which the decision variable has a lower bound and an upper bound. Infinite dimensional quadratic programming programs on measure spaces were proposed in Wu [12]. In that paper Wu provided a cutting plane approach to solving quadratic infinite programs on measure spaces. In this paper, we study infinite dimensional quadratic programming problems in the L_p space where $1<p<\infty$, and we require that the decision variable in the L_p space where $1<p<\infty$ has a lower bound and an upper bound on a compact interval. These types of problems are related to Vanderbei's study. Here, we also review [1,2,4,7,10] for our research of this paper.

In the following, $L_q(X)$, $1<q<\infty$ and $1/p+1/q=1$, is considered as the primal space. Thus, $L_p(X)$, $1<p<\infty$ and $1/p+1/q=1$, is the dual space of $L_q(X)$. In this situation, $L_q(X)$ is a separable Banach space, and therefore any weak* compact subset of $L_p(X)$ in the weak* topology is metrizable from the result of Theorem 3.16 in Rudin [9]. Consequently, any weak* compact subset of $L_p(X)$ is sequentially compact in the weak* topology.

Now, we state a proposition which is useful for this paper as follows:

Proposition 1.1. *Suppose that $f\in C(X\times X)$. If for any $k\in L_p(X)$ and every sequence $\{k_n\}$ such that $\lim_{n\to\infty}k_n=k$ in the weak* topology, then we have*:

$$\lim_{n\to\infty}\int_X\int_X f(s,t)k_n(s)\,\mathrm{d}s k_n(t)\,\mathrm{d}t=\int_X\int_X f(s,t)k(s)\,\mathrm{d}s k(t)\,\mathrm{d}t.$$

The proof of Proposition 1.1 mainly applies basic ideas of uniform continuity and uniform convergence, so we omit the proof. Here, we denote by F the feasible set of (P). By the second constraint of (P), there exists an $M>0$ such that $\|k\|_{L_p}\leqslant M$ for each $k\in F$. Hence, F is bounded in the L_p-norm. We define the set B_M as follows:

$$B_M=\{k\in L_p(X):\|k\|_{L_p}\leqslant M\}.$$

Note that the set B_M is weak* compact in the weak* topology. Then we have the following theorem.

Theorem 1.1. *Suppose that $F\neq\emptyset$. Then (P) has an optimal solution.*

Proof. Since the primal space $L_q(X)$ is a separable Banach space, B_M is metrizable. Let k be in the weak* closure of F. Note that $F\subset B_M$. There exists a sequence $\{k_i\}\subset F$ such that $\lim_{i\to\infty}k_i=k$ in the weak* topology. Since $\{k_i\}\subset F$, it follows that $\int_X\phi(s,y)k_i(s)\,\mathrm{d}s\geqslant g(y)$ for each $y\in Y$. Here, we consider $\phi\in C(X\times Y)$. Hence, $\phi(s,y)\in C(X)$ for each fixed $y\in Y$. Consequently, $\phi(s,y)\in L_q(X)$ for each fixed $y\in Y$. Applying $\lim_{i\to\infty}k_i=k$ in the weak* topology, we have $\lim_{i\to\infty}\int_X\phi(s,y)k_i(s)\,\mathrm{d}s=\int_X\phi(s,y)k(s)\,\mathrm{d}s$ for each $y\in Y$. Hence, the inequalities $\int_X\phi(s,y)k(s)\,\mathrm{d}s\geqslant g(y)$ for each $y\in Y$ follow.

Now we want to prove that $M_1\leqslant k(s)\leqslant M_2$ a.e. on X and we will do so by contradiction. There are two cases which may occur.

Case 1: There would exist a measurable subset $A\subset X$ of Lebesgue measure greater than 0, such that $k(s)<M_1$ for each $s\in A$.

Case 2: There would exist a measurable subset $B\subset X$ of Lebesgue measure greater than 0, such that $k(s)>M_2$ for each $s\in B$.

First we deal with Case 1. We denote the Lebesgue measure of A by $L(A)$. Then the characteristic function χ_A is in $L_q(X)$. Thus, we have

$$\int_X\chi_A(s)k(s)\,\mathrm{d}s=\lim_{i\to\infty}\int_X\chi_A(s)k_i(s)\,\mathrm{d}s. \tag{1}$$

This implies that

$$\int_A k(s)\,\mathrm{d}s=\lim_{i\to\infty}\int_A k_i(s)\,\mathrm{d}s\geqslant\lim_{i\to\infty}\int_A M_1\,\mathrm{d}s=M_1L(A). \tag{2}$$

From the definition of A, it follows that