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# Recursive computation of Hermite spherical spline interpolants ☆

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**Abstract**

Let $u$ be a function defined on a spherical triangulation $\Delta$ of the unit sphere $S$. In this paper, we study a recursive method for the construction of a Hermite spline interpolant $u_k$ of class $\mathscr{C}^k$ and degree $4k+1$ on $S$, defined by some data scheme $D_k(u)$. We show that when the data sets $D_r(u)$ are nested, i.e., $D_{r-1}(u) \subset D_r(u)$, $1 \leqslant r \leqslant k$, the spline function $u_k$ can be decomposed as a sum of $k+1$ simple elements. This decomposition leads to the construction of a new and interesting basis of a space of Hermite spherical splines. The theoretical results are illustrated by some numerical examples.


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## 1. Introduction

The well-known methods for building the classical univariate or bivariate Hermite spline interpolants are based on the Hermite fundamental functions. But, the lack of recursive formulae for computing these basis functions makes this construction rather complicated. In order to overcome this difficulty, a simple method allowing to compute recursively a univariate Hermite spline interpolant of class $\mathscr{C}^k$ and degree $2k+1$ of a function $f$ defined on an interval $[a, b]$ was proposed in [9] (see also [10]). More precisely, if $f_k$ is such an interpolant, then it can be decomposed in the form $f_k = f_0 + g_1 + \cdots + g_k$, where $f_0$ is the piecewise linear interpolant of $f$, and $g_r$, $1 \leqslant r \leqslant k$, are particular splines of $\mathscr{C}^{r-1}$ and degree $2r+1$ that satisfy interesting properties. The simplicity and the multiresolution structure of this decomposition make it attractive for applications, such as computing integrals, smoothing curves and compressing data. For more details on these subjects, see [9,10].

In view of the importance and the originality of this method, it is natural to extend it to several variables. One obvious way to do this is to use the tensor product. With regard to this extension, a recursive construction for tensor product Hermite interpolants was described in [7]. In [11] (see also [8], a method allowing to build recursively bivariate Hermite spline interpolants of class $\mathscr{C}^k$ on $\mathbb{R}^2$ was proposed. In this paper, we deal with a hierarchical computation of particular $\mathscr{C}^k$ Hermite spherical spline interpolants.

Assume that $S$ is the unit sphere, and $V = \{v_i\}_{i=1}^n$ is a set of scattered points located on $S$. Let us denote by $\Delta$ a spherical triangulation of $S$ whose set of vertices is $V$. For a regular function $u$, defined on $S$, we denote by $D_k(u)$ the

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set of data formed by the values and the derivatives of $u$ at all the vertices $v_i$ of $\Delta$ and at other points lying in $S$. Let $D_{k,T}$ be the data set in $D_k(u)$ restricted to a spherical triangle $T$ of $\Delta$. We show that there exists a unique spherical Bernstein–Bézier (SBB) polynomial $u_{k,T}$ of degree $4k+1$ defined on $T$ which interpolates the data $D_{k,T}(u)$. Setting $u_k = \sum_{T\in\Delta} u_{k,T}$, then $u_k$ is the unique spherical spline of smoothness $k$ and degree $4k+1$ which interpolates $D_k(u)$. Methods of this type are called macro-element methods.

Our aim in this paper is to define a recursive formula allowing to compute $u_k$ step by step if some conditions are satisfied. In order to do this, assume that the sets $D_{r,T}(u)$, $0 \leqslant r \leqslant k$, are nested, i.e.,

$$D_{0,T} \subset D_{1,T} \subset \cdots \subset D_{k,T} \quad \text{for all } T \in \Delta. \tag{1.1}$$

Then, the Hermite spherical spline $u_k$ can be written in the form $u_k = u_0 + d_1 + \cdots + d_k$, where $u_0$ is the interpolant to $u$ of class $\mathscr{C}^0$ and degree 1 and $d_r$, $1 \leqslant r \leqslant k$, is a spherical spline of class $\mathscr{C}^{r-1}$ and degree $4r+1$ on $S$. If we put $d_{r,T} = d_r|_T$, then we have $u_{k,T} = u_{0,T} + d_{1,T} + \cdots + d_{k,T}$, so $u_0 = \sum_{T\in\Delta} u_{0,T}$ and $d_r = \sum_{T\in\Delta} d_{r,T}$, $1 \leqslant r \leqslant k$. Moreover, each $d_{r,T}$, which is an homogeneous Bernstein–Bézier polynomial, is completely determined by the data $D_{k,T}(u - u_r)$, $1 \leqslant r \leqslant k$. The multiresolution structure of this decomposition means that $u_0$ may be considered as a coarse approximation of $u_k$, and $d_r$, $0 \leqslant r \leqslant k$, are correction terms or detail functions. This representation of $u_k$ gives rise to a family that generates the space of spherical splines of smoothness $k$ and degree $4k+1$, and to a new basis for the space $B_{4k+1}(T)$, $T \in \Delta$, of homogeneous Bernstein–Bézier polynomials of degree $4k+1$. As a consequence of (1.1), we will see later that the new bases for the spaces $B_{4r+1}(T)$, $0 \leqslant r \leqslant k$, are hierarchical. Then, they can be used as tools for solving several mathematic problems like those studied in [4].

The paper is organized as follows. In Section 2 we give some preliminary results on homogeneous Bernstein–Bézier polynomials and spherical splines. Section 3 is devoted to local interpolation method based on $C^k$ macro-elements of degree $4k+1$. In Section 4 we define a recursive computation of local Hermite polynomials $u_{k,T} \in B_{4k+1}(T)$, $T \in \Delta$, when their corresponding data schemes $D_{k,T}(u)$ are nested. Then we deduce a decomposition of the spherical spline $u_k$ of class $\mathscr{C}^k$ and degree $4k+1$ on $S$. As a consequence of this method, we obtain a new and interesting basis for $B_{4k+1}(T)$. Finally, in order to illustrate our results, we give in Section 5 some numerical examples.

## 2. Preliminary results

In this section, we present the connection between the functions defined on $S$ and homogeneous trivariate functions, and we introduce some definitions.

A trivariate function $F$ is said to be positively homogeneous of degree $t \in \mathbb{R}$ provided that for every real number $a > 0$,

$$F(av) = a^t F(v), \quad v \in \mathbb{R}^3 \backslash \{0\}.$$

**Lemma 1** (*see Alfeld et al. [3]*). *Given a function f defined on S and let $t \in \mathbb{R}$. Then*

$$F_t(v) = \|v\|^t f\left(\frac{v}{\|v\|}\right)$$

*is the unique homogeneous extension of f of degree t to all of $\mathbb{R}^3 \backslash \{0\}$, i.e., $F_t|_S = f$, and $F_t$ is homogeneous of degree t.*

Let $g$ be a given unit vector. Then, as in [3], we define the directional derivative $D_g$ of $f$ at a point $v \in S$ by

$$D_g f(v) = D_g F(v) = g^{\mathrm{T}} \nabla F(v),$$

where $F$ is some homogeneous extension of $f$ and $\nabla F$ is the gradient of the trivariate function $F$.

While a polynomial of degree $d$ has a natural homogeneous extension to $\mathbb{R}^3$, a general function $f$ on $S$ has infinitely many different extensions. The value of its derivative may depend on which extension that we take (for more details see [3]).

Let $\mathscr{P}_d$ be the space of trivariate polynomials of total degree at most $d$, and let $\mathscr{H}_d = \mathscr{P}_d|_S$ be its restriction to the sphere $S$. A trivariate polynomial $p$ is called homogeneous of degree $d$ if $p(\lambda x, \lambda y, \lambda z) = \lambda^d p(x, y, z)$ for all $\lambda \in \mathbb{R}$, and harmonic if $\Delta p = 0$, where $\Delta$ is the Laplace operator defined by $\Delta f = (D_x^2 + D_y^2 + D_z^2) f$.