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The RKGL method for the numerical solution of initial-value problems

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Abstract

We introduce the RKGL method for the numerical solution of initial-value problems of the form $y' = f(x, y)$, $y(a) = \alpha$. The method is a straightforward modification of a classical explicit Runge–Kutta (RK) method, into which Gauss–Legendre (GL) quadrature has been incorporated. The idea is to enhance the efficiency of the method by reducing the number of times the derivative $f(x, y)$ needs to be computed. The incorporation of GL quadrature serves to enhance the global order of the method by, relative to the underlying RK method. Indeed, the RKGL method has a global error of the form $Ah^{r+1} + Bh^{2m}$, where r is the order of the RK method and m is the number of nodes used in the GL component. In this paper we derive this error expression and show that RKGL is consistent, convergent and strongly stable.

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1. Introduction

The numerical solution to the initial-value problem (IVP)

$$y' = f(x, y), \quad y(a) = \alpha \tag{1}$$

on some interval $[a, b]$ is often obtained using a Runge–Kutta (RK) method [3,4]. These methods are consistent, convergent, stable and are easily programmed, and, as such, are usually the method of choice for problems as in (1). It is true, however, that RK methods of high order (more accurate) require greater computational effort [2]. In this paper we describe a straightforward modification to a classical explicit RK method, designed to improve the efficiency of the method. The resulting method is designated the RKGL method, or RKrGLm (this notation will become clear later).

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2. Terminology, notation and relevant concepts

Here we describe notation, terminology and concepts relevant to the rest of the paper.

- We denote an explicit RK method by

$$y_{i+1} = y_i + hF(x_i, y_i). \tag{2}$$

For example, if we have

$$k_1 = f(x, y), \quad k_2 = f\left(x + \frac{h}{2}, y + \frac{hk_1}{2}\right), \tag{3}$$

$$k_3 = f\left(x + \frac{h}{2}, y + \frac{hk_2}{2}\right), \quad k_4 = f(x + h, y + hk_3), \tag{4}$$

then

$$F(x, y) = \frac{k_1}{6} + \frac{k_2}{3} + \frac{k_3}{3} + \frac{k_4}{6} \tag{5}$$

corresponds to the classical fourth-order RK method.
- The true value of y at x_i is denoted by $y(x_i)$ and the approximate value of y at x_i is denoted by y_i.
- The global error Δ_i in y_i is defined by

$$y_i = y(x_i) + \Delta_i. \tag{6}$$

- Gauss–Legendre (GL) quadrature on the interval $[-1, 1]$ is given by [1,6]

$$\int_{-1}^{1} f(x)\,\mathrm{d}x \approx \sum_{i=1}^{m} W_i f(x_i). \tag{7}$$

Here, there are m nodes on the interval $[-1, 1]$, and W_i are appropriate weights. On an arbitrary interval GL quadrature is

$$\int_{u}^{v} f(x), \mathrm{d}x \approx \frac{(v-u)}{2} \sum_{i=1}^{m} W_i f(\tilde{x}_i) = h \sum_{i=1}^{m} \widehat{W}_i^m f(\tilde{x}_i), \tag{8}$$

where $\widehat{W}_i^m \doteqdot (m+1)W_i/2$, and h denotes the average length of the subintervals into which $[u, v]$ is subdivided by the nodes $\tilde{x}_i$. We have used the symbol $\tilde{x}_i$ for the nodes on $[u, v]$ to differentiate from the nodes x_i on $[-1, 1]$; indeed, $\tilde{x}_i = (u + v + (v - u)x_i)/2$. However, in the remainder of this paper x_i will be used as a generic symbol for the nodes.
- GL quadrature on an interval using m nodes is denoted by GLm.
- It is a simple matter to show that the error in GLm quadrature is $\mathrm{O}(h^{2m+1})$ (see Appendix).
- The RK method of global order r is denoted by RKr. Such a method has global error $\mathrm{O}(h^r)$ and local error $\mathrm{O}(h^{r+1})$.
- The method denoted by RKrGLm is a method involving RKr and GLm.
- The parameter p is defined as $p \doteqdot m + 1$.

3. The RKGL method

In this section we describe the RKGL algorithm. To begin with, consider a subinterval of $[a, b]$ on which discrete nodes $\{a = x_0, x_1, \ldots, x_m\}$ have been defined. We use RKr to find a solution at the m nodes $\{x_1, \ldots, x_m\}$. The solution