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Superlinearly convergent PCG algorithms for some nonsymmetric elliptic systems☆

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Abstract

A preconditioned conjugate gradient method is applied to finite element discretizations of some nonsymmetric elliptic systems. Mesh independent superlinear convergence is proved, which is an extension of a similar earlier result from a single equation to systems. The proposed preconditioning method involves decoupled preconditioners, which yields small and parallelizable auxiliary problems.

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Keywords: Conjugate gradient method; Nonsymmetric elliptic systems; Superlinear convergence; Mesh independence

1. Introduction

The conjugate gradient method (CGM) is a widespread way of solving nonsymmetric linear algebraic systems, in particular for large systems arising from discretized elliptic problems. A celebrated property of the CGM is superlinear convergence [17], see also the book [2] where a comprehensive summary is given on the convergence of the CGM. For discretized elliptic problems, the CGM is mostly used with suitable preconditioning [2], which often provides mesh independent convergence. Moreover, it has been shown in [6] that the preconditioned CGM can be competitive with multigrid methods.

The mesh independence property is a basic reason to involve underlying Hilbert space theory in the study of the CGM. Linear convergence results for such PCG methods are treated in the rigorously described framework of equivalent operators in Hilbert space [6,13], which provides mesh independence for the condition numbers of the discretized problems. The CGM for nonsymmetric equations in Hilbert space has been studied in the author's papers [3,4]: in the latter superlinear convergence has been proved in Hilbert space and, based on this, mesh independence of the superlinear estimate has been derived for FEM discretizations of elliptic Dirichlet problems. The numerical realization of this method has been demonstrated in [11].

The goal of this paper is to extend the mesh independent superlinear convergence results of [4] from a single equation to systems. An important advantage of the obtained preconditioning method for systems is that one can define

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decoupled preconditioners, hence the size of the auxiliary systems remains as small as for a single equation, moreover, parallelization of the auxiliary systems is available.

2. The problem and the approach

We consider systems of the form

$$\left.\begin{array}{l} -\mathrm{div}\,(K_i \nabla u_i) + \mathbf{b}_i \cdot \nabla u_i + \sum_{j=1}^{l} V_{ij} u_j = g_i \\ u_{i\,|\partial\Omega} = 0 \end{array}\right\} \quad (i = 1, \ldots, l) \tag{1}$$

under the following assumptions:

Assumptions BVP.

(i) the bounded domain $\Omega \subset \mathbf{R}^N$ is C^2-diffeomorphic to a convex domain;
(ii) for all $i, j = 1, \ldots, l$ $\;K_i \in C^1(\overline{\Omega})$, $\;V_{ij} \in L^\infty(\Omega)$ and $\mathbf{b}_i \in C^1(\overline{\Omega})^N$;
(iii) there is $m > 0$ such that $K_i \geqslant m$ holds for all $i = 1, \ldots, l$;
(iv) letting $V = \{V_{ij}\}_{i,j=1}^{l}$, the coercivity property

$$\lambda_{\min}(V + V^{\mathrm{T}}) - \max_i \mathrm{div}\ \mathbf{b}_i \geqslant 0 \tag{2}$$

holds pointwise on Ω, where $\lambda_{\min}$ denotes the smallest eigenvalue;
(v) $g_i \in L^2(\Omega)$.

The coercivity assumption implies that problem (1) has a unique weak solution.

Systems of the form (1) arise e.g., from the time discretization and Newton linearization of nonlinear reaction–convection–diffusion (transport) systems

$$\left.\begin{array}{l} \dfrac{\partial c_i}{\partial t} - \mathrm{div}\,(K_i\, \nabla c_i) + \mathbf{b}_i \cdot \nabla c_i + R_i(x, c_1, \ldots, c_l) = 0 \\ c_{i\,|\partial\Omega} = 0 \end{array}\right\} \quad (i = 1, \ldots, l). \tag{3}$$

In many real-life problems, e.g., where c_i are concentrations of chemical species, such systems may consist of a huge number of equations [18]. Using a time discretization with sufficiently small steplength τ, the systems obtained from the Newton linearization of (3) around some $\mathbf{c} = (c_1, \ldots, c_l)^{\mathrm{T}}$ satisfy Assumptions BVP. Namely, in this case

$$V(x) = \frac{\partial R(x, \mathbf{c})}{\partial \mathbf{c}} + \frac{1}{\tau}\mathbf{I}$$

(where $\mathbf{I}$ is the identity matrix), which ensures the coercivity (the only nontrivial assumption) for small enough τ.

For brevity, we write (1) as

$$\left.\begin{array}{l} L\mathbf{u} \equiv -\mathrm{div}\,(\mathbf{K}\,\nabla \mathbf{u}) + \mathbf{b} \cdot \nabla \mathbf{u} + V\mathbf{u} = \mathbf{g} \\ \mathbf{u}_{|\partial\Omega} = \mathbf{0} \end{array}\right\}, \tag{4}$$

where

$$\mathbf{u} := \begin{pmatrix} u_1 \\ \vdots \\ u_l \end{pmatrix}, \quad \mathbf{g} := \begin{pmatrix} g_1 \\ \vdots \\ g_l \end{pmatrix},$$

$$-\mathrm{div}\,(\mathbf{K}\nabla\mathbf{u}) := \begin{pmatrix} -\mathrm{div}\,(K_1 \nabla u_1) \\ \vdots \\ -\mathrm{div}\,(K_l \nabla u_l) \end{pmatrix}, \quad \mathbf{b} \cdot \nabla\mathbf{u} := \begin{pmatrix} \mathbf{b}_1 \cdot \nabla u_1 \\ \vdots \\ \mathbf{b}_l \cdot \nabla u_l \end{pmatrix}$$