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Sturm–Liouville problems with parameter dependent potential and boundary conditions

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Abstract

This paper deals with the computation of the eigenvalues of Sturm–Liouville problems with parameter dependent potential and boundary conditions. We shall extend the domain of application of the method based on sampling theory to the case where the classical Whittaker–Shannon–Kotel'nikov theorem is not applicable. A few numerical examples will be presented.

MSC: 34L15; 35C10

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1. Introduction

We shall extend the method based on sampling theory [1] (see also [2–4]) to compute the eigenvalues of Sturm–Liouville problems with parameter dependent potential and boundary conditions (for motivations, see for example [5,6,10,11], for more on sampling theory see [12]). Consider the Sturm–Liouville problem

$$\begin{cases} -y'' + q(x,\mu)y = \mu^2 y, \quad x \in [x_0, x_1], \\ A(y(x_0), y'(x_0), y(x_1), y'(x_1))^{\mathrm{T}} = 0, \end{cases} \tag{1.1}$$

where $q(x,\mu) = q_0(x) + q_1(x)\mu + q_2(x)\mu^2$, q_0, q_1, q_2 are complex valued functions belonging to $L^1(x_0, x_1)$, the matrix

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \end{pmatrix}$$

has rank 2, and the a_{ij} are functions of μ.

Note that if the right-hand side of the differential equation is of the form $\mu^2 w(x)y$ we can just add $\mu^2(1 - w(x))y$ to both sides of the differential equation and include the $w(x)$ term in $q_2(x)$. This fact will allow us avoid using

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the Liouville transformation [7,8,13] to eliminate the $w(x)$ term. Note also that this will allow us tackle indefinite Sturm–Liouville problems as particular cases of this class of problems.

We introduce as usual $y_c(x,\mu)$ and $y_s(x,\mu)$ solutions off the base problems

$$\begin{cases} -y''+q(x,\mu)y=\mu^2 y, & x\in(x_0,x_1],\\ y(x_0)=1, \quad y'(x_0)=0 \end{cases} \tag{1.2}$$

and

$$\begin{cases} -y''+q(x,\mu)y=\mu^2 y, & x\in(x_0,x_1],\\ y(x_0)=0, \quad y'(x_0)=1, \end{cases} \tag{1.3}$$

respectively.

It is well known that if the potential q is independent of μ, $y_s(x,\mu)$ and $y_c(x,\mu)-\cos(\mu(x-x_0))$ belong to the Paley–Wiener space PW_σ where $\sigma=x_1-x_0$ and

$$\mathrm{PW}_\sigma=\{f \text{ entire}, |f(z)|\leqslant c\exp[\sigma|\mathrm{Im}\, z|],\ \ f\in L^2(R)\}.$$

The sampling method consists in recovering $y_s(x_1,\mu)$ and $y_c(x_1,\mu)-\cos(\mu(x_1-x_0))$ using the well-known WSK theorem.

Theorem 1 (*Whittaker–Shannon–Kotel'nikov, Zayed [12]*). *Let $f\in PW_\sigma$ then*

$$f(\mu)=\sum_{k=-\infty}^{\infty} f\left(\frac{k\pi}{\sigma}\right)\frac{\sin\sigma(\mu-k\pi/\sigma)}{\sigma(\mu-k\pi/\sigma)}, \tag{1.4}$$

where the series converges uniformly on compact subsets of IR and also in $L^2_{\mathrm{d}\mu}$.

We shall consider in the remaining of the paper the case where the potential q depends on μ.

2. Main result

In what follows we shall need the known estimates.

Lemma 2. $|\sin z/z|\leqslant\beta_1 \mathrm{e}^{|\mathrm{Im}\, z|}/(1+|z|)$ *and* $|\cos z|\leqslant \mathrm{e}^{|\mathrm{Im}\, z|}$ *where β_1 is a positive constant* (*we may take $\beta_1=1.72$*).

We claim the following:

Theorem 3. *$y_c(x,\mu)$, $y_s(x,\mu)$, $y_c'(x,\mu)$ and $y_s'(x,\mu)$ are entire as functions of μ for each fixed $x\in(x_0,x_1]$ and satisfy the growth conditions*

$$\begin{aligned}
&|y_c(x,\mu)|, \quad |y_s(x,\mu)|, \quad |y_c(x,\mu)-\cos(\mu(x-x_0))|,\\
&\left|y_s(x,\mu)-\frac{\sin(\mu(x-x_0))}{\mu}\right|,\\
&|y_c'(x,\mu)+\mu\sin(\mu(x-x_0))|, \quad |y_s'(x,\mu)-\cos(\mu(x-x_0))|\leqslant\beta_0 \mathrm{e}^{\alpha(x)|\mu|}
\end{aligned}$$

for some positive constant β_0 and $\alpha(x)=\beta_1\int_{x_0}^{x}|q_2(\xi)|\,\mathrm{d}\xi+(x-x_0)+\varepsilon$, ε being a small positive number.

Proof. First we transform (1.2) and (1.3) into the integral equations

$$y_s(x,\mu)=\frac{\sin\mu(x-x_0)}{\mu}+\int_{x_0}^{x}\frac{\sin\mu(x-\tau)}{\mu}q(\tau,\mu)y_s(\tau,\mu)\,\mathrm{d}\tau \tag{2.1}$$