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Beyond the Poisson renewal process: A tutorial survey

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Abstract

After sketching the basic principles of renewal theory and recalling the classical Poisson process, we discuss two renewal processes characterized by waiting time laws with the same power asymptotics defined by special functions of Mittag–Leffler and of Wright type. We compare these three processes with each other.

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1. Introduction

The Poisson process is well known to play a fundamental role in renewal theory. In the present paper, by using special functions of Mittag–Leffler and Wright type in the definitions of waiting time distributions, we provide a generalization and a variant to this classical process. These distributions, characterized by power-law asymptotics in contrast to the exponential law of the Poisson process, have been studied by several authors both from mathematical and physical point of view, see e.g., [2,10–12,14,21] and references therein.

The structure of our paper is as follows. In Section 2, we recall the basic renewal theory including its fundamental concepts like waiting time between events, the survival probability, the renewal function. If the waiting time is exponentially distributed we have the classical Poisson process, which is Markovian: this is the topic of Section 3. However, other waiting time distributions are also relevant in applications, in particular such ones having a power-law decay of their density. In this context we analyse, respectively, in Sections 4 and 5, two non-Markovian renewal processes with waiting time distributions described by functions of Mittag–Leffler and Wright type: both depend on a parameter $\beta \in (0, 1)$ related to the common exponent in the power law. In the limit $\beta = 1$ the first becomes the Poisson process whereas the second goes over into the deterministic process producing its events at equidistant instants of time. In Section 6, after sketching the differences between the renewal processes of Mittag–Leffler and Wright type, we compare numerically their survival functions and their probability densities in the special case $\beta = \frac{1}{2}$ with respect to the corresponding functions of the classical Poisson process. Concluding remarks are given in Section 7.

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2. Essentials of renewal theory

We present a brief introduction to the renewal theory by using our notation. For more details see e.g., the classical treatises by Cox [3], Feller [5], and the more recent book by Ross [23]. We begin to recall that a stochastic process $\{N(t),\ t \geqslant 0\}$ is called a *counting process* if $N(t)$ represents the total number of "events" that have occurred up to time t. It is called a *renewal process* if the times between successive events, $T_1, T_2, \ldots,$ are independent identically distributed (*iid*) non-negative random variables, obeying a given probability law. We call these times *waiting times* (or inter-arrival times) and the instants $t_0 = 0$, $t_k = \sum_{j=1}^{k} T_j\ (k = 1, 2, \ldots)$ *renewal times*. Let the *waiting times* be distributed like T and let

$$\Phi(t) := P(T \leqslant t) \tag{2.1}$$

be the common probability distribution function, that we assume to be absolutely continuous. Then the corresponding probability density function[1] $\phi(t)$ and the probability distribution function $\Phi(t)$ are related by

$$\phi(t) = \frac{\mathrm{d}}{\mathrm{d}t}\Phi(t), \quad \Phi(t) = \int_0^t \phi(t')\,\mathrm{d}t'. \tag{2.2}$$

We recall that $\phi(t) \geqslant 0$ with $\int_0^\infty \phi(t)\,\mathrm{d}t = 1$ and $\Phi(t)$ is a non-decreasing function in $\mathbf{R}^+$ with $\Phi(0) = 0$, $\Phi(+\infty) = 1$. Often, especially in Physics, the *probability density function* is abbreviated by *pdf*, so that, in order to avoid confusion, the probability distribution function is called the *probability cumulative function* and abbreviated by *pcf*. When the non-negative random variable represents the lifetime of a technical system, it is common practice to call $\Phi(t)$ the *failure probability* and

$$\Psi(t) := P(T > t) = \int_t^\infty \phi(t')\,\mathrm{d}t' = 1 - \Phi(t), \tag{2.3}$$

the *survival probability*, because $\Phi(t)$ and $\Psi(t)$ are the respective probabilities that the system does or does not fail in $(0, t]$. These terms, however, are commonly adopted for any renewal process.

As a matter of fact, the *renewal process* is defined by the *counting process*

$$N(t) := \begin{cases} 0 & \text{for } 0 \leqslant t < t_1, \\ \max\{k | t_k \leqslant t,\ k = 1, 2, \ldots\} & \text{for } t \geqslant t_1. \end{cases} \tag{2.4}$$

$N(t)$ is thus the random number of renewals occurring in $(0, t]$. We easily recognize that $\Psi(t) = P(N(t) = 0)$. Continuing in the general theory, we set $F_1(t) = \Phi(t)$, $f_1(t) = \phi(t)$, and in general

$$F_k(t) := P(t_k = T_1 + \cdots + T_k \leqslant t), \quad f_k(t) = \frac{\mathrm{d}}{\mathrm{d}t} F_k(t), \quad k \geqslant 1. \tag{2.5}$$

$F_k(t)$ is the probability that the sum of the first k waiting times does not exceed t, and $f_k(t)$ is the corresponding density. $F_k(t)$ is normalized because $\lim_{t\to\infty} F_k(t) = P(t_k = T_1 + \cdots + T_k < \infty) = \Phi(+\infty) = 1$. In fact, the sum of k random variables each of which is finite with probability 1 is finite with probability 1 itself. We set for consistency $F_0(t) = \Theta(t)$, the Heaviside unit step function (with $\Theta(0) := \Theta(0^+)$) so that $F_0(t) \equiv 1$ for $t \geqslant 0$, and $f_0(t) = \delta(t)$, the Dirac delta generalized function.

A relevant quantity is the function $v_k(t)$ that represents the probability that k events occur in the interval $(0, t]$. We get, for any $k \geqslant 0$,

$$v_k(t) := P(N(t) = k) = P(t_k \leqslant t, t_{k+1} > t) = \int_0^t f_k(t')\Psi(t - t')\,\mathrm{d}t'. \tag{2.6}$$

For $k = 0$ we recover $v_0(t) = \Psi(t)$.

[1] Let us remark that, as it is popular in Physics, we use the word density also for generalized functions that can be interpreted as probability measures. In these cases the function $\Phi(t)$ may lose its absolute continuity.