Available online at www.sciencedirect.com
ScienceDirect
Journal of Computational and Applied Mathematics 205 (2007) 849–858
JOURNAL OF COMPUTATIONAL AND APPLIED MATHEMATICS
www.elsevier.com/locate/cam

Numerical modelling of qualitative behaviour of solutions to convolution integral equations

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Received 25 April 2005

Abstract

We consider the qualitative behaviour of solutions to linear integral equations of the form

$$y(t) = g(t) + \int_0^t k(t-s)y(s)\,\mathrm{d}s, \tag{1}$$

where the kernel k is assumed to be either integrable or of exponential type. After a brief review of the well-known Paley–Wiener theory we give conditions that guarantee that exact and approximate solutions of (1) are of a specific exponential type. As an example, we provide an analysis of the qualitative behaviour of both exact and approximate solutions of a singular Volterra equation with infinitely many solutions. We show that the approximations of neighbouring solutions exhibit the correct qualitative behaviour.

MSC: 45E10; 65R20; 45M99

Keywords: Integral equations; Qualitative behaviour; Resolvent kernels; Numerical methods

1. Introduction

We are concerned with preserving qualitative properties of exact solutions of equations under discrete (numerical) approximations. These qualitative features include (but are not restricted to) asymptotic behaviour of the solution set to a problem for large time. Much of the existing literature is concerned with solutions that converge to zero as $t \to \infty$ or which are bounded as $t \to \infty$. However, it can be equally important to consider the effectiveness of numerical schemes in reproducing other types of behaviour. Thus, in recent work we have been concerned [7,6,9–12] with periodic or oscillatory behaviour of solutions, and with the preservation of bifurcation points in numerical solutions of delay and integral equations.

This paper is concerned with equations where we are able to deduce results about *rates of growth or decay*. We develop analytical results for the exact solutions of some convolution integral equations and for their numerical approximations and we show how our results provide insights into the exact and numerical solution of some singular integral equations of interest from the literature.

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0377-0427/$ - see front matter
doi:10.1016/j.cam.2006.04.061

For linear equations, such as (1) there can be a close correspondence between qualitative behaviour of solutions and stability of a particular solution under small perturbations of the problem. For example, consider the perturbation of Eq. (1) caused by changing the forcing function g to $\tilde{g}$. (We assume the restriction that $g, \tilde{g} \in \mathscr{G}$, some linear space of permitted (*admissible*) forcing functions.) The difference in the two solutions now satisfies an equation of form (1) with the forcing function g replaced by $\tilde{g} - g \in \mathscr{G}$. Now if all solutions to (1) for $g \in \mathscr{G}$ are bounded or tend to zero, then so will be the perturbation caused by the change in g. Thus, if all solutions to (1) are bounded, then all will be stable with respect to changes in the forcing term; if all tend to zero for large t then all will be asymptotically stable with respect to perturbations of this type.

2. Preliminaries

As is customary, we shall use the notation $L^p(\mathbb{R}^+)$ to represent the class of Lebesgue measurable functions for which $\int_0^\infty |f(t)|^p \, dt < \infty$. In practice, it may be more appropriate to restrict our functions to have a finite number of discontinuities. The class $BC(\mathbb{R}^+)$ will be used for the bounded continuous functions on the positive real line and the class $PC(\mathbb{R}^+)$ will refer to the set of piecewise continuous functions.

For our discussion of qualitative behaviour of solutions (see the next section) it will be convenient to introduce the idea of *functions of exponential type*.

Definition 2.1. A function f is of exponential order as $t \to \infty$ if there exist constants A, α such that $|f(t)| \leqslant A e^{\alpha t}$ for all $t \geqslant 0$. A measurable function f is of exponential type if there exists some constant α for which $f(t)e^{-\alpha t} \in L^1(\mathbb{R}^+)$. A measurable function f is of exponential type σ if there exists a constant σ for which $f(t)e^{-\sigma t} \in L^1(\mathbb{R}^+)$.

Remark 2.1. Note that if we define $s = \inf\{\sigma: f \text{ is of exponential type } \sigma\}$ then f need not necessarily be of exponential type s.

Definition 2.2. For two functions x, y defined on $\mathbb{R}^+$ the convolution product is defined by

$$x * y(t) = \int_0^t x(t-s)y(s) \, ds \tag{2}$$

for every value of $t \geqslant 0$ for which the integral exists.

This provides us with convenient notation that enables us to write Eq. (1) in the form

$$y(t) = g(t) + k * y(t). \tag{3}$$

For measurable functions f of exponential type σ, the Laplace transform, denoted

$$\mathscr{L}[f](s) = \int_0^\infty e^{-st} f(t) \, dt \tag{4}$$

exists and is finite for $\Re(s) \geqslant \sigma$.

Properties of the convolution that will be of interest in this paper are contained in the following theorem and corollary. Further details can be found in [4,13].

Theorem 2.1. (1) $\mathscr{L}[x * y](s) = \mathscr{L}[x](s)\mathscr{L}[y](s)$ *providing all the Laplace transforms exist.*

(2) *If* $x \in L^1(\mathbb{R}^+)$ *and* $y \in L^p(\mathbb{R}^+)$ *then* $x * y \in L^p(\mathbb{R}^+)$.

Corollary 2.1. (1) *If* x, y *are measurable functions of exponential type* σ *then* $x * y$ *is of exponential type* σ.

(2) *If* x, y *are bounded continuous functions then* $x * y$ *is of exponential order*.

Remark 2.2. Part (i) of the corollary can be proved by direct substitution into the definition of convolution and part (ii) follows from part (i) by considering all values of $\sigma > 0$.

For some of our work, it turns out that the property of positive definiteness will be important. We will use the following important characterisation of positive definite functions, due to Bochner.