Available online at www.sciencedirect.com
ScienceDirect
Journal of Computational and Applied Mathematics 201 (2007) 3 – 7
JOURNAL OF COMPUTATIONAL AND APPLIED MATHEMATICS
www.elsevier.com/locate/cam

Letter to the Editor

Note on the complex zeros of $H'_\nu(x) + \mathrm{i}\zeta H_\nu(x) = 0$

Sven-Erik Sandström*, Christian Ackrén

Department of Mathematics and Systems Engineering, University of Växjö, S-351 95, Växjö, Sweden

Received 13 September 2005; received in revised form 17 January 2006

Abstract

Approximations for the complex zeros in the ν-plane of the Hankel functions $H_\nu(x)$ and $H'_\nu(x)$ are available in terms of the zeros of Airy functions. The corresponding zeros of a linear combination of Hankel functions are of interest when scattering from cylinders with a surface impedance is studied. A simple method to compute these zeros efficiently is presented.

MSC: 33C10; 34E05; 78A45

Keywords: Zeros of special functions; Asymptotic expansions; Iterative solution

1. Introduction

The zeros of the Hankel functions in the complex ν-plane relate to scattering from circular cylinders [2,3] and also to the Regge poles of quantum mechanics. One important function of these zeros is to determine the propagation constants of creeping waves [9,10]. Expressions for the zeros for the two basic scattering problems (two polarizations) can be found in Ref. [3]. A more difficult problem arises when an impedance boundary condition is used. One then looks for the zeros, in the complex variable corresponding to the index, of a linear combination of the Hankel function and its derivative. This problem was solved by Bouche et al. [2] by approximating the Hankel functions in terms of Airy functions and then applying a numerical procedure to the resulting approximate problem. Other combinations of numerical procedures, suitable for small values of x, have also been devised [4]. This note presents an approximate solution to the original problem.

2. Discussion

The zeros of $H_\nu(x)$ and $H'_\nu(x)$ are given in terms of the parameter x that corresponds to frequency, the zeros of the Airy function and its derivative $a_s = -\alpha_s$, $a'_s = -\beta_s$ [1,3] and the Fock parameter $m = (x/2)^{1/3}$. Asymptotic expansions

* Corresponding author. Tel.: +46 470 708135.
E-mail address: sesan@msi.vxu.se (S.-E. Sandström).

0377-0427/$ - see front matter
doi:10.1016/j.cam.2006.01.032

will be used and the value of x; and hence also of m, is therefore assumed to be large. Approximate zeros are given by

$$v_s = x + e^{i\pi/3}\alpha_s m - e^{-i\pi/3}\frac{\alpha_s^2}{60}m^{-1} - \frac{1}{140}\left(1 - \frac{\alpha_s^3}{10}\right)m^{-3}, \tag{1}$$

$$\bar{v}_s = x + e^{i\pi/3}\beta_s m - \frac{e^{-i\pi/3}}{10}\left(\beta_s^{-1} + \frac{1}{6}\beta_s^2\right)m^{-1} + \frac{1}{200}\left(\beta_s^{-3} + 4 + \frac{\beta_s^3}{7}\right)m^{-3}. \tag{2}$$

The combination of Hankel functions that is studied in this note,

$$H_v'(x) + i\zeta H_v(x) = 0, \tag{3}$$

involves the complex parameter ζ that corresponds to either a surface impedance or a surface admittance depending on the polarization of the field [2]. We refer to the function $H_v^{(1)}(x)$ [1]. Also in this case, the zeros can be found by extending the approach devised by Franz and Galle [6].

Asymptotic expansions for the Hankel function and its derivative are obtained from Watson's integral representation by means of the saddle-point method. From these expansions, in terms of Airy functions, the zeros in Eqs. (1) and (2) were extracted by means of Taylor expansions of the Airy functions, in the neighbourhood of their zeros [6]. By simply combining the expansions one obtains approximate zeros, corresponding to the impedance ζ, in terms of the zeros of either the Airy function or its derivative.

$$v_{s\zeta} = x + e^{i\pi/3}\beta_s m + i\zeta e^{-i\pi/3}m^2\left(\frac{1}{\beta_s} - \frac{e^{i\pi/3}}{15m^2}\right) - \frac{e^{-i\pi/3}}{10m}\left(\frac{1}{\beta_s} + \frac{\beta_s^2}{6}\right), \quad m|\zeta| < 1, \tag{4}$$

$$v_{s\zeta} = x + e^{i\pi/3}\alpha_s m + \frac{1}{i\zeta}\left(1 + \frac{e^{i\pi/3}\alpha_s}{15m^2}\right) - \frac{e^{-i\pi/3}\alpha_s^2}{60m}, \quad m|\zeta| > 1. \tag{5}$$

These expressions are fairly accurate for the first zeros, i.e., small values of the index s. The error with respect to the exact zero is larger when $m|\zeta|$ is close to 1 since neither of the Taylor expansions is accurate when the linear combination does not clearly emphasize one of the functions, cf. Eq. (3). Similar results were derived by Keller et al. [7] from a linear combination of Airy functions with constant coefficients, under the assumption that $m|\zeta|$ is either large or small. Cochran [5] uses uniform expansions systematically to derive higher-order terms, but with no significant improvement in accuracy. Streifer [11] also derives higher order terms for large and small $m|\zeta|$ but the method has the same deficiency for $m|\zeta| \approx 1$. Fig. 1 shows the difference between the approximate zero and the exact zero as a

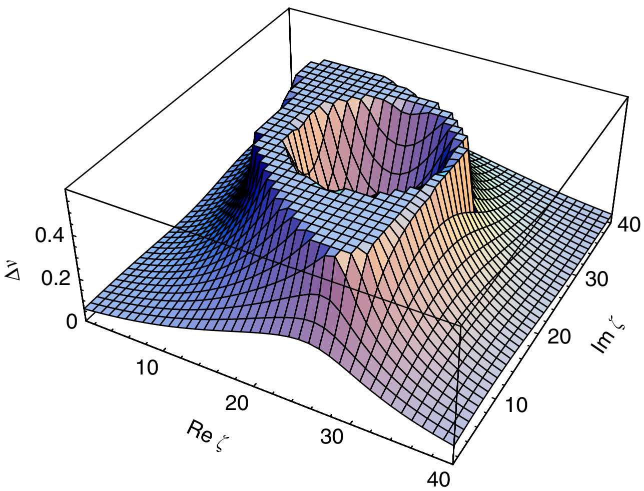

Fig. 1. The error $\Delta v = |v_{s\zeta} - v_{se}|$, as a function of the complex surface impedance ζ, computed by means of asymptotic expansions of the Hankel function and Taylor expansions of the Airy function: $x = 10$, $s = 1$. A grid that corresponds to $-1.3 < \mathrm{Re}\,\zeta < 1.3$, $-1.3 < \mathrm{Im}\,\zeta < 1.3$ is used. The surface is truncated at about half the maximum value for the sake of clarity. The circle $m|\zeta| = 1$ can be discerned in the graph.