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Mechanical quadrature methods and their extrapolation for solving first kind Abel integral equations☆

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Abstract

This paper presents high accuracy mechanical quadrature methods for solving first kind Abel integral equations. To avoid the ill-posedness of problem, the first kind Abel integral equation is transformed to the second kind Volterra integral equation with a continuous kernel and a smooth right-hand side term expressed by weakly singular integrals. By using periodization method and modified trapezoidal integration rule, not only high accuracy approximation of the kernel and the right-hand side term can be easily computed, but also two quadrature algorithms for solving first kind Abel integral equations are proposed, which have the high accuracy $O(h^2)$ and asymptotic expansion of the errors. Then by means of Richardson extrapolation, an approximation with higher accuracy order $O(h^3)$ is obtained. Moreover, an a posteriori error estimate for the algorithms is derived. Some numerical results show the efficiency of our methods.

Keywords: The first kind Abel integral equation; Quadrature method; Asymptotic expansion; Extrapolation

1. Introduction

The first kind Abel integral equation

$$\int_0^x \frac{H(x,y)}{(x-y)^\alpha} f(y)\,\mathrm{d}y = g(x) \quad (0 \leqslant x \leqslant 1,\ 0<\alpha<1) \tag{1.1}$$

frequently appears in many physical and engineering problems, e.g., semi-conductors, scattering theory, seismology, heat conduction, metallurgy, fluid flow, chemical reactions and population dynamics, etc. (see [12]). If $H(x,y) = 1/\Gamma(1-\alpha)$, (1.1) can be written in the form

$$J^{1-\alpha} f = \frac{1}{\Gamma(1-\alpha)} \int_0^x \frac{1}{(x-y)^\alpha} f(y)\,\mathrm{d}y = g(x), \tag{1.2}$$

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which is called the fractional integral equation of order $1-\alpha$. There is a vast literature on fractional calculus and their applications, e.g., Gorenflo [9,10] and Gorenflo and Mainardi [11]. By means of fractional derivative, the solution of (1.2) can be expressed as

$$f(x)=D^{1-\alpha}g(x)=\frac{1}{\Gamma(\alpha)}\frac{\mathrm{d}}{\mathrm{d}x}\int_0^x\frac{g(y)}{(x-y)^{1-\alpha}}\,\mathrm{d}y=\frac{1}{\Gamma(\alpha)}\int_0^x\frac{g'(y)}{(x-y)^{1-\alpha}}\,\mathrm{d}y+\frac{g(0^+)}{\Gamma(\alpha)}x^{\alpha-1}, \tag{1.3}$$

which implies that $f(x)$ is unbounded at $x=0$ if $g(0)\neq 0$. Moreover if $g(0)=0$, the derivative $f'(x)$ of the solution $f(x)$ of (1.1) still may be unbounded at the origin (see [2]). These properties bring on much difficulty for the numerical treatment of (1.1).

There are many classes of numerical methods that have been developed over past few decades for the approximate solution of Eq. (1.1), such as product-integration methods [1,7], collocation methods [4–6], backward Euler methods [8,10] and fractional multistep methods [22], etc. Unfortunately, the first kind Abel integral equation (1.1) is often regarded as an ill-posed problem. The numerical treatment is more difficult for first kind Abel integral equations than for second kind ones, which have been widely studied (see, e.g., [1,2,15,16,21]). For (1.2), Gorenflo [9,10] presented some numerical methods of the fractional calculus, e.g., the Grünwald–Letnikov difference approximation

$$D_h^{1-\alpha}f(x)=h^{\alpha-1}(\delta_h^{1-\alpha}g)(x)=h^{\alpha-1}\sum_{j=0}^{[x/h+(1-\alpha)/2]}(-1)^j\binom{1-\alpha}{j}g(x+((1-\alpha)/2-j)h). \tag{1.4}$$

The formula (1.4) has accuracy order $O(h^2)$ if $g(x)$ is sufficiently smooth and vanish at $x\leqslant 0$, else has accuracy order $O(h)$.

On the other hand, Lubich [14] introduced a fractional multistep method for Abel–Volterra integral equation of the first kind, and Plato [22] considered fractional multistep methods for weakly singular Volterra integral equations of the first kind with perturbed right-hand side. In this paper, we propose a completely different approach for solving (1.1).

Unlike the first kind Fredholm integral equation, which essentially is an ill-posed problem, the first kind Abel integral equation can be transformed into the second kind Abel integral equation by a standard method (see [1]). If we replace x by s in (1.1), multiply both sides by $(x-s)_+^{\alpha-1}$, and then integrate with respect to s, (1.1) can be written as

$$\int_0^x L(x,y)f(y)\,\mathrm{d}y=G(x), \tag{1.5}$$

where

$$L(x,y)=\int_y^x\frac{H(s,y)}{(x-s)^{1-\alpha}(s-y)^{\alpha}}\,\mathrm{d}s=\int_0^1\frac{H(y+\tau(x-y),y)}{(1-\tau)^{1-\alpha}\tau^{\alpha}}\,\mathrm{d}\tau, \tag{1.6}$$

and

$$G(x)=\int_0^x\frac{g(y)}{(x-y)^{1-\alpha}}\,\mathrm{d}y=x^{\alpha}\int_0^1\frac{g(x\tau)}{(1-\tau)^{1-\alpha}}\,\mathrm{d}\tau. \tag{1.7}$$

Since $L(x,x)=H(x,x)\pi/\sin(\pi\alpha)\neq 0$ for $0\leqslant x\leqslant 1$ and $G(0)=0$, differentiating (1.5) with respect to x, we get

$$f(x)+\int_0^x\tilde{L}(x,y)f(y)\,\mathrm{d}y=v(x)\quad(0\leqslant x\leqslant 1), \tag{1.8}$$

where $\tilde{L}(x,y)=L_x(x,y)/L(x,x)$ and $v(x)=G'(x)/L(x,x)$. Thus, (1.1) is transformed into a second kind Volterra integral equation, whose kernel and the right-hand side term are expressed by weakly singular integrals.

Since the solution $f(x)$ of (1.1) or its derivative $f'(x)$ may be unbounded at the origin, Baratella and Orsi [2] proposed to take the change of variable $x=\gamma(t)=t^q$ in (1.8), where q is an undetermined positive constant. Then (1.8) is written as

$$f(\gamma(t))+\int_0^{\gamma(t)}\tilde{L}(\gamma(t),y)f(y)\,\mathrm{d}y=v(\gamma(t))\quad(0\leqslant t\leqslant 1).$$