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Sequences of orthogonal Laurent polynomials, bi-orthogonality and quadrature formulas on the unit circle[☆]

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Abstract

In this paper, the construction of orthogonal bases in the space of Laurent polynomials on the unit circle is considered. As an application, a connection with the so-called bi-orthogonal systems of trigonometric polynomials is established and quadrature formulas on the unit circle based on Laurent polynomials are studied.

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1. Introduction

Throughout the paper, we shall be dealing with a positive Borel measure μ supported on the unit circle $\mathbb{T} := \{z \in \mathbb{C} : |z| = 1\}$ and concerned with the computation of integrals on $\mathbb{T}$, i.e., integrals of the form

$$I_\mu(f) = \int_{-\pi}^{\pi} f(\mathrm{e}^{\mathrm{i}\theta})\,\mathrm{d}\mu(\theta). \tag{1.1}$$

As usual, estimations of $I_\mu(f)$ will be produced when replacing in (1.1) $f(z)$ by an appropriate approximating (interpolating) function $L(z)$ so that $I_\mu(L)$ can now be easily computed. The space Λ of Laurent polynomials consists of all functions of the form $L(z) = \sum_{j=p}^{q} \alpha_j z^j$, $\alpha_j \in \mathbb{C}$, $p, q \in \mathbb{Z}$, $p \leqslant q$. Because of the density of Λ in $C(\mathbb{T}) = \{f : \mathbb{T} \to \mathbb{C},\ f\ continuous\}$ with respect to the uniform norm (see e.g. [9, pp. 304–305]) it seems reasonable to choose as an approximation to $f(z)$ in (1.1) some appropriate Laurent polynomial. For further approximation problems on $\mathbb{T}$ see [19]. In this respect, the so-called "quadrature formulas on the unit circle", or "Szegö formulas" introduced in [16] (see also [22]) appear as the analoges on $\mathbb{T}$ of the Gauss–Christoffel or Gaussian formulas when dealing with the estimation of weighted integrals over intervals $[a, b]$ $(-\infty \leqslant a < b \leqslant +\infty)$ on the real line. Here, it should be recalled (see e.g. [12]) the fundamental role played by the orthogonal polynomials as an orthogonal basis of Π, the space of the

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ordinary polynomials. In this respect, the initial aim of the paper will be the construction of an orthogonal basis for Λ with respect to the inner product induced by $\mu(\theta)$, i.e.

$$\langle f, g\rangle_\mu = \int_{-\pi}^{\pi} f(\mathrm{e}^{\mathrm{i}\theta})\overline{g(\mathrm{e}^{\mathrm{i}\theta})}\,\mathrm{d}\mu(\theta) \tag{1.2}$$

and the application of these bases in the estimation of $I_\mu(f)$.

The paper has been organized as follows. In Section 2, some preliminary results concerning quadrature rules on the unit circle will be briefly given. The construction of orthogonal bases in Λ will be made in Section 3, while the connection with bi-orthogonal systems of trigonometric polynomials is shown in Section 4. The quadrature formulas are analyzed in Section 5, and finally, some concluding remarks emphasizing the role played by Szegö polynomials will be presented in Section 6.

The following notations will be used. Namely,

$$\mathbb{D} := \{z : |z| < 1\}, \quad \overline{\mathbb{D}} := \mathbb{D} \cup \mathbb{T}, \quad \mathbb{E} := \{z : |z| > 1\}, \quad \overline{\mathbb{E}} := \mathbb{E} \cup \mathbb{T}$$

$$\Lambda_{p,q} = \operatorname{span}\{z^j : p \leqslant j \leqslant q\}, \quad p, q \in \mathbb{Z}, \quad p \leqslant q, \quad \Pi_n = \{z^j : 0 \leqslant j \leqslant n\} = \Lambda_{0,n}.$$

Also, for a given function $f(z)$ we define the "substar-conjugate" as $f_*(z) = \overline{f(1/\overline{z})}$ and for a polynomial $P(z)$ of degree n its reciprocal (or reverse) $P^*(z) = z^n P_*(z) = z^n \overline{P(1/\overline{z})}$. If $L(z) = \sum_{j=p}^{q} \alpha_j z^j \in \Lambda_{p,q}$ then the "complex-conjugate" is defined as $\overline{L}(z) = \sum_{j=p}^{q} \overline{\alpha_j} z^j \in \Lambda_{p,q}$. Moreover, for all $k \in \mathbb{Z}$ we set $\mu_k = \int_{-\pi}^{\pi} \mathrm{e}^{-\mathrm{i}k\theta}\,\mathrm{d}\mu(\theta)$ (kth trigonometric moment) and denote by Δ_n the nth Toeplitz determinant for the measure μ, i.e.

$$\Delta_n = \begin{vmatrix} \mu_0 & \cdots & \mu_n \\ \vdots & \ddots & \vdots \\ \mu_{-n} & \cdots & \mu_0 \end{vmatrix} > 0, \quad n = 0, 1, 2, \ldots .$$

Furthermore, $\{\rho_n(z)\}_{n=0}^{\infty}$ will denote the sequence of monic orthogonal polynomials on the unit circle for $\mu(\theta)$ (Szegö polynomials). This means that for each $n \geqslant 1$, $\rho_n(z)$ is a monic polynomial of exact degree n satisfying

$$\langle \rho_n(z), z^s\rangle_\mu = \langle \rho_n^*(z), z^t\rangle_\mu = 0, \quad s = 0, 1, \ldots, n-1, \quad t = 1, 2, \ldots, n,$$

$$\langle \rho_n(z), z^n\rangle_\mu = \langle \rho_n^*(z), 1\rangle_\mu = \frac{\Delta_n}{\Delta_{n-1}} > 0. \tag{1.3}$$

It should be noted that, in general, explicit expressions for Szegö polynomials are not available, and if we want to compute them we can make use of the following (Szegö) recurrence relations (see e.g. [13,23]):

$$\rho_0(z) = \rho_0^*(z) = 1,$$

$$\rho_n(z) = z\rho_{n-1}(z) + \delta_n \rho_{n-1}^*(z), \quad n = 1, 2, 3, \ldots,$$

$$\rho_n^*(z) = \overline{\delta_n} z \rho_{n-1}(z) + \rho_{n-1}^*(z), \quad n = 1, 2, 3, \ldots, \tag{1.4}$$

where $\delta_n := \rho_n(0)$ for all $n = 1, 2, \ldots$ are the so-called *Schur parameters* with respect to $\mu(\theta)$.[2] They satisfy $|\delta_n| < 1$ for $n \geqslant 1$ since the zeros of $\rho_n(z)$ lie in $\mathbb{D}$ (see [21]). Now, we introduce the useful sequence of nonnegative real numbers $\{\eta_n\}_{n=1}^{\infty}$ by

$$\eta_n = \sqrt{1 - |\delta_n|^2} \in (0, 1], \quad n = 1, 2, \ldots . \tag{1.5}$$

Finally, observe that from the recurrence relations (1.4) it follows for $n = 1, 2, \ldots$ that

$$\rho_{n-1}^*(z) = \frac{1}{\eta_n^2}[\rho_n^*(z) - \overline{\delta_n}\rho_n(z)] \tag{1.6}$$

[2] There are at least four other terms: *Szegö parameters*, *reflection parameters*, *Verblunsky parameters* or *Geronimus parameters* (see [21]).