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A class of logarithmically completely monotonic functions and the best bounds in the first Kershaw's double inequality

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Abstract

In the article, the logarithmically complete monotonicity of a class of functions involving Euler's gamma function are proved, a class of the first Kershaw-type double inequalities are established, and the first Kershaw's double inequality and Wendel's inequality are generalized, refined or extended. Moreover, an open problem is posed.

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1. Introduction

It is well known that the classical Euler's gamma function Γ can be defined for $x>0$ as $\Gamma(x)=\int_0^\infty e^{-t}t^{x-1}\,dt$. The digamma or psi function ψ is defined as the logarithmic derivative of Γ and $\psi^{(i)}$ for $i\in\mathbb{N}$ are called polygamma functions.

The ratio $\Gamma(s)/\Gamma(r)$ has been researched by many mathematicians in the past more than fifty years. Wendel [30] gave for $0<b<1$ and $x>0$ the following double inequality:

$$\left(\frac{x}{x+b}\right)^{1-b}\leqslant\frac{\Gamma(x+b)}{x^b\Gamma(x)}\leqslant 1. \tag{1}$$

Gautschi showed in [8] for $0<s<1$ and $n\in\mathbb{N}$ that

$$n^{1-s}<\frac{\Gamma(n+1)}{\Gamma(n+s)}<\exp[(1-s)\psi(n+1)]. \tag{2}$$

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A strengthened upper bound was given by Erber in [7]:

$$\frac{\Gamma(n+1)}{\Gamma(n+s)} < \frac{4(n+s)(n+1)^{1-s}}{4n+(s+1)^2}, \quad 0<s<1, \quad n \in \mathbb{N}. \tag{3}$$

Kečkić and Vasić gave in [12] the inequalities below:

$$\frac{b^{b-1}}{a^{a-1}} \cdot e^{a-b} < \frac{\Gamma(b)}{\Gamma(a)} < \frac{b^{b-1/2}}{a^{a-1/2}} \cdot e^{a-b}, \quad 0<a<b. \tag{4}$$

The following closer bounds were proved for $0<s<1$ and $x \geqslant 1$ by Kershaw in [13]:

$$\left(x+\frac{s}{2}\right)^{1-s} < \frac{\Gamma(x+1)}{\Gamma(x+s)} < \left[x-\frac{1}{2}+\left(s+\frac{1}{4}\right)^{1/2}\right]^{1-s}, \tag{5}$$

$$\exp[(1-s)\psi(x+s^{1/2})] < \frac{\Gamma(x+1)}{\Gamma(x+s)} < \exp\left[(1-s)\psi\left(x+\frac{s+1}{2}\right)\right]. \tag{6}$$

Let s and t be nonnegative numbers, $\alpha=\min\{s,t\}$, and

$$z_{s,t}(x) = \begin{cases} \left[\dfrac{\Gamma(x+t)}{\Gamma(x+s)}\right]^{1/(t-s)} - x, & s \neq t, \\ e^{\psi(x+s)} - x, & s = t \end{cases} \tag{7}$$

in $x \in (-\alpha, \infty)$. In [5,6,27], a monotonicity and convexity of $z_{s,t}(x)$ was obtained: The function $z_{s,t}(x)$ is either convex and decreasing for $|t-s|<1$ or concave and increasing for $|t-s|>1$. From this, the best bounds in the first Kershaw's double inequality (5) were deduced.

For a and b being two constants, as $x \to \infty$, the following asymptotic formula is given in [1, pp. 257, 259]:

$$x^{b-a}\frac{\Gamma(x+a)}{\Gamma(x+b)} = 1 + \frac{(a-b)(a+b-1)}{2x} + \mathrm{O}\left(\frac{1}{x^2}\right). \tag{8}$$

For recent development and more detailed information on this topic, please refer to, for example, [5,6,16,27] and the references therein.

Recall [18,31] that a function f is said to be completely monotonic on an interval I if f has derivatives of all orders on I and $(-1)^n f^{(n)}(x) \geqslant 0$ for $x \in I$ and $n \geqslant 0$. Recall [23–25] also that a function f is called logarithmically completely monotonic on an interval I if f has derivatives of all orders on I and its logarithm $\ln f$ satisfies $0 \leqslant (-1)^k [\ln f(x)]^{(k)}$ for all $k \in \mathbb{N}$ on I. For our own convenience, the sets of the completely monotonic functions and the logarithmically completely monotonic functions on I are denoted, respectively, by $\mathscr{C}[I]$ and $\mathscr{L}[I]$. In [3,18,23–26], it has been proved that $\mathscr{L}[I] \subset \mathscr{C}[I]$. The well-known Bernstein's Theorem [31, p. 161] states that $f \in \mathscr{C}[(0,\infty)]$ if and only if there exists a bounded and nondecreasing function $\eta(t)$ such that the integral $f(x)=\int_0^\infty e^{-xt}\,\mathrm{d}\eta(t)$ converges for $0<x<\infty$. In [3, Theorem 1.1, 9] it is pointed out that the logarithmically completely monotonic functions on $(0,\infty)$ can be characterized as the infinitely divisible completely monotonic functions studied by Horn in [11, Theorem 4.4]. For more information on the classes $\mathscr{C}[I]$ and $\mathscr{L}[I]$, please refer to [2,3,9,18,20,23–27,29] and the references therein.

For $x>0$ and $a>0$, let

$$h_a(x) = \frac{(x+a)^{1-a}\Gamma(x+a)}{x\Gamma(x)} \quad \text{and} \quad f_a(x) = \frac{\Gamma(x+a)}{x^a\Gamma(x)}, \tag{9}$$

where Γ is the classical Euler's gamma function. In [21,22], among other things, the logarithmically completely monotonic properties of the functions $h_a(x)$ and $f_a(x)$ are obtained:

(1) $\lim_{x\to 0+} h_a(x) = \Gamma(a+1)/a^a$ and $\lim_{x\to\infty} h_a(x) = 1$ for any $a>0$;
(2) $h_a(x) \in \mathscr{L}[(0,\infty)]$ if $0<a<1$;
(3) $[h_a(x)]^{-1} \in \mathscr{L}[(0,\infty)]$ if $a>1$;
(4) $\lim_{x\to\infty} f_a(x) = 1$ for any $a \in (0,\infty)$;