Available online at www.sciencedirect.com
ScienceDirect
Journal of Computational and Applied Mathematics 208 (2007) 3 – 19
JOURNAL OF COMPUTATIONAL AND APPLIED MATHEMATICS
www.elsevier.com/locate/cam

The Bessel differential equation and the Hankel transform

W.N. Everitt[a,*], H. Kalf[b]

[a] School of Mathematics and Statistics, University of Birmingham, Edgbaston, Birmingham B15 2TT, England, UK
[b] Mathematisches Institut, Universität München, Theresienstrasse 39, D-80333 München, Germany

Received 31 August 2005

Dedicated to Professor W.D. Evans on the occasion of his 65th birthday

Abstract

This paper studies the classical second-order Bessel differential equation in Liouville form:

$$-y''(x) + (v^2 - \tfrac{1}{4})x^{-2}y(x) = \lambda y(x) \quad \text{for all } x \in (0, \infty).$$

Here, the parameter v represents the order of the associated Bessel functions and λ is the complex spectral parameter involved in considering properties of the equation in the Hilbert function space $L^2(0, \infty)$.

Properties of the equation are considered when the order $v \in [0, 1)$; in this case the singular end-point 0 is in the limit-circle non-oscillatory classification in the space $L^2(0, \infty)$; the equation is in the strong limit-point and Dirichlet condition at the end-point $+\infty$.

Applying the generalised initial value theorem at the singular end-point 0 allows of the definition of a single Titchmarsh–Weyl m-coefficient for the whole interval $(0, \infty)$. In turn this information yields a proof of the Hankel transform as an eigenfunction expansion for the case when $v \in [0, 1)$, a result which is not available in the existing literature.

The application of the principal solution, from the end-point 0 of the Bessel equation, as a boundary condition function yields the Friedrichs self-adjoint extension in $L^2(0, \infty)$; the domain of this extension has many special known properties, of which new proofs are presented.
© 2006 Elsevier B.V. All rights reserved.

MSC: primary 34B24; 34B30; 33C10; secondary 34L05; 33C05

Keywords: Bessel differential equation; Titchmarsh–Weyl m-coefficient; Hankel transform; Friedrichs extension

1. Introduction

In this paper we consider the Bessel differential equation in the classical form

$$-y''(x) + (v^2 - 1/4)x^{-2}y(x) = \lambda y(x) \quad \text{for all } x \in (0, \infty). \tag{1.1}$$

Here, the parameter $v \in [0, \infty) \subset \mathbb{R}$ is the order of the Bessel functions involved, and the parameter $\lambda \in \mathbb{C}$ is the spectral parameter. Properties of this equation are considered in the Hilbert function space $L^2(0, \infty)$.

* Corresponding author. Tel.: +44 121 414 6587; fax: +44 121 414 3389.
E-mail addresses: w.n.everitt@bham.ac.uk (W.N. Everitt), hubert.kalf@mathematik.uni-muenchen.de (H. Kalf).

0377-0427/$ - see front matter © 2006 Elsevier B.V. All rights reserved.
doi:10.1016/j.cam.2006.10.029

We restrict attention to the situation when $\nu \in [0, 1)$; in this case the endpoint 0 of Eq. (1.1) is in the singular limit-circle case, with respect to $L^2(0, \infty)$, except for the regular case when $\nu = \frac{1}{2}$.

For a regular endpoint a Sturm–Liouville equation, such as (1.1) when $\nu = \frac{1}{2}$, there is a classical solution to the initial value problem which yields the analytic dependence of the solutions on the complex spectral parameter λ; see for example [19, Chapter I, Section 1.5]. Recent studies have shown that for a limit-circle endpoint there is a generalised solution to the initial value problem, which reduces to the classical solution when the endpoint is regular; see [7, Sections 1–5, in particular Theorem 2; 1, Section 5, Theorem 5.1].

This generalised solution, to the initial value problem, allows for the definition of the Titchmarsh–Weyl m-coefficient associated with a singular boundary condition at the limit-circle endpoint; see details in the recent paper [1, Section 8]. From the Nevanlinna representation of this m-coefficient the spectral function ρ can be obtained to describe the spectrum of the associated self-adjoint operator in $L^2(0, \infty)$.

By choosing the self-adjoint operator to be the Friedrichs extension, see [15,16,8,13,17], it then proves possible to obtain the Hankel transform formula, see [18, Chapter VIII, Section 8.18; 19, Chapter IV, Section 4.11], as an eigenfunction expansion, even in the case when $\nu \in [0, 1)$.

Additional analysis then yields the limit behaviour of the functions in the domain of the Friedrichs extension, as previously discussed by a number of authors [11,17]. These results allow for discussion of a possible HELP-type integral inequality as previously considered for regular endpoints in [5,6].

We have made reference in the text to the earlier work of other authors whose papers are listed in the References.

2. Bessel differential equation

For the differential equation (1.1) on the interval $(0, \infty)$ we restrict attention to the case when the order parameter $\nu \in [0, 1)$; this restriction has the implications (a) and (b) below for the endpoint classifications at 0 and ∞ in the space $L^2(0, \infty)$; for general details of these classifications see [5, Section 3; 6, Section 5].

We use the following named Bessel functions, see [20, Chapter III], as solutions of (1.1):

(i) For $\nu = 0$

$$x^{1/2} J_0(x\sqrt{\lambda}) \quad \text{and} \quad x^{1/2} Y_0(x\sqrt{\lambda}) \ \text{ for all } x \in (0, \infty). \tag{2.1}$$

(ii) For $\nu \in (0, 1)$

$$x^{1/2} J_\nu(x\sqrt{\lambda}) \quad \text{and} \quad x^{1/2} J_{-\nu}(x\sqrt{\lambda}) \ \text{ for all } x \in (0, \infty). \tag{2.2}$$

Here and after the analytic function $\sqrt{\cdot} : \mathbb{C} \to \mathbb{C}$ is defined as follows:

$$\text{given } \lambda = \rho \exp(\mathrm{i}\eta) \quad \text{define } \sqrt{\lambda} := \rho^{1/2} \exp(\tfrac{1}{2}\,\mathrm{i}\eta) \ \text{ for } \rho \in [0, \infty) \ \text{ and } \ \eta \in [0, 2\pi). \tag{2.3}$$

In the solution $x^{1/2} Y_0(x\sqrt{\lambda})$ there is a term $\log(\frac{1}{2}\, x\sqrt{\lambda})$ which here is defined by, using again $\lambda = \rho \exp(\mathrm{i}\eta)$,

$$\log(\tfrac{1}{2}\, x\sqrt{\lambda}) := \ln(\tfrac{1}{2}\, x\rho^{1/2}) + \mathrm{i}\,\tfrac{1}{2}\,\eta. \tag{2.4}$$

It is clear from the properties of solutions (2.1) and (2.2), see [5, Section 3; 20, Chapter III], that:

(a) The endpoint $+\infty$ is strong limit-point and Dirichlet in the space $L^2(0, \infty)$; see [6, Section 5] for details of these properties.
(b) The endpoint 0^+ is limit-circle non-oscillatory (for $\nu = \frac{1}{2}$ this endpoint is regular but this classification may be regarded as limit-circle non-oscillatory; we make only infrequent special mention of this exception).

3. Hankel transform

Formally the Hankel inversion formula (here also called the Hankel transform) can be written in symmetrical form as

$$f(x) = \int_0^\infty (xs)^{1/2} J_\nu(xs)\,\mathrm{d}s \int_0^\infty (s\xi)^{1/2} J_\nu(s\xi) f(\xi)\,\mathrm{d}\xi \quad \text{for all } x \in (0, \infty). \tag{3.1}$$

A systematic account of the various forms in which this transform can be considered is given in [18, Chapter VIII].

Download English Version:

https://daneshyari.com/en/article/4642574

Download Persian Version:

https://daneshyari.com/article/4642574

Daneshyari.com