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CMV matrices: Five years after☆

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Abstract

CMV matrices are the unitary analog of Jacobi matrices; we review their general theory.

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1. Introduction

The Arnold Principle: If a notion bears a personal name, then this name is not the name of the inventor.

The Berry Principle: The Arnold Principle is applicable to itself. *V.I. Arnold*, *On Teaching Mathematics*, 1997 [8] (Arnold says that Berry formulated these principles.)

In 1848, Jacobi [45] initiated the study of quadratic forms $J(x_1, \ldots, x_n) = \sum_{k=1}^{n} b_k x_k^2 + 2\sum_{k=1}^{n-1} a_k x_k x_{k+1}$, that is, essentially $n \times n$ matrices of the form

$$J = \begin{pmatrix} b_1 & a_1 & 0 & \ldots & 0 \\ a_1 & b_2 & a_2 & \ldots & 0 \\ 0 & a_2 & b_3 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \ldots & \ldots & a_{n-1} & b_n \end{pmatrix} \tag{1.1}$$

and found that the eigenvalues of J were the zeros of the denominator of the continued fraction

$$\cfrac{1}{b_1 - z - \cfrac{a_1^2}{b_2 - z - \cfrac{a_2^2}{\cdots}}}. \tag{1.2}$$

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In the era of the birth of the spectral theorem, Toeplitz [79], Hellinger–Toeplitz [44], and especially Stone [75] realized that Jacobi matrices were universal models of self-adjoint operators, A, with a cyclic vector, φ_0.

To avoid technicalities, consider the case where A is bounded, and suppose initially that $\mathscr{H}$ is infinite-dimensional. By cyclicity, $\{A^k\varphi_0\}_{k=0}^\infty$ are linearly independent, so by applying Gram–Schmidt to $\varphi_0, A\varphi_0, A^2\varphi_0, \ldots,$ we get polynomials $p_j(A)$ of degree exactly j with positive leading coefficients so that

$$\varphi_j = p_j(A)\varphi_0 \tag{1.3}$$

are an orthonormal basis for $\mathscr{H}$. By construction,

$$\varphi_j \perp \varphi_0, A\varphi_0, \ldots, A^{j-1}\varphi_0$$

so

$$\langle \varphi_j, A\varphi_k\rangle = 0, \quad j \geqslant k+2. \tag{1.4}$$

Because A is self-adjoint, we see $\langle \varphi_j, A\varphi_k\rangle = 0$ also if $j \leqslant k-2$. Thus, the matrix $\langle \varphi_j, A\varphi_k\rangle$ has exactly form (1.1) where $a_j > 0$ (since $p_j(A)$ has leading positive coefficient).

Put differently, for all A, φ_0, there is a unitary $U : \mathscr{H} \to \ell^2$ (given by Fourier components in the φ_j basis), so UAU^{-1} has the form J and $\varphi_0 = (1, 0, 0, \ldots)^{\mathrm{t}}$. The Jacobi parameters, $\{a_n, b_n\}_{n=1}^\infty$, are intrinsic, which shows there is exactly one J (with $\varphi_0 = (1, 0, 0, \ldots)^{\mathrm{t}}$) in the unitary equivalence class of (A, φ_0).

There is, of course, another way of describing unitary invariants for (A, φ_0): the spectral measure $\mathrm{d}\mu$ defined by

$$\int x^n \, \mathrm{d}\mu(x) = \langle \varphi_0, A^n\varphi_0\rangle. \tag{1.5}$$

There is a direct link from $\mathrm{d}\mu$ to the Jacobi parameters: the $p_j(x)$ are orthonormal polynomials associated to $\mathrm{d}\mu$, and the Jacobi parameters are associated to the three-term recursion relation obeyed by the p's:

$$xp_j(x) = a_{j+1}p_{j+1} + b_{j+1}p_j(x) + a_j p_{j-1}(x) \tag{1.6}$$

(where $p_{-1} \equiv 0$).

Here we are interested in the analog of these structures for unitary matrices. We begin by remarking that for a general normal operator, N, the right form of cyclicity is that $\{N^k(N^*)^\ell\varphi_0\}_{k,\ell=0}^\infty$ is total. Since $A = A^*$, only $\{A^k\varphi_0\}_{k=0}^\infty$ enters. Since $U^* = U^{-1}$, for unitaries $U^k(U^*)^\ell = U^{k-\ell}$ and the right notion of cyclicity is that $\{U^k\varphi_0\}_{k=-\infty}^\infty$ is total.

Some parts of the above fourfold equivalence:

(1) unitary equivalence classes of (A, φ_0);
(2) spectral measures, that is, probability measures $\mathrm{d}\mu$ on $\mathbb{R}$ with bounded support and infinite support;
(3) Jacobi parameters;
(4) Jacobi matrices

are immediate for the unitary case. Namely, (1) $\Leftrightarrow$ (2) holds since there is a spectral theorem for unitaries, and so, a one–one correspondence between unitary equivalence classes of (U, φ_0) on infinite-dimensional spaces and probability measures on $\partial\mathbb{D}$ ($\mathbb{D} = \{z \mid |z| < 1\}$) with infinite support.

More subtle is the analog of Jacobi parameters. Starting from such a probability measure on $\partial\mathbb{D}$, one can form the monic orthogonal polynomials $\Phi_n(z)$ and find (see [77]; see also [69, Section 1.5]) $\{\alpha_n\}_{n=0}^\infty \in \mathbb{D}^\infty$, so

$$z\Phi_n(z) = \Phi_{n+1}(z) + \bar{\alpha}_n z^n \, \overline{\Phi_n(1/\bar{z})}. \tag{1.7}$$

While Verblunsky [81] defined the α_n in a different (albeit equivalent) way, he proved a theorem (called Verblusnky's theorem in [69]; see also [68]) that says this map $\mathrm{d}\mu \to \{\alpha_n\}_{n=0}^\infty$ is one–one and onto all of $\mathbb{D}^\infty$, so (1)–(3) for the unitary case have been well understood for 65 years.