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ScienceDirect
Journal of Computational and Applied Mathematics 208 (2007) 164–175
JOURNAL OF COMPUTATIONAL AND APPLIED MATHEMATICS
www.elsevier.com/locate/cam

A continuum of unusual self-adjoint linear partial differential operators

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Received 7 October 2005

Dedicated to Professor W.D. Evans on the occasion of his 65th birthday

Abstract

In an earlier publication a linear operator T_{Har} was defined as an unusual self-adjoint extension generated by each linear elliptic partial differential expression, satisfying suitable conditions on a bounded region Ω of some Euclidean space. In this present work the authors define an extensive class of T_{Har}-like self-adjoint operators on the Hilbert function space $L_2(\Omega)$; but here for brevity we restrict the development to the classical Laplacian differential expression, with Ω now the planar unit disk. It is demonstrated that there exists a non-denumerable set of such T_{Har}-like operators (each a self-adjoint extension generated by the Laplacian), each of which has a domain in $L_2(\Omega)$ that does not lie within the usual Sobolev Hilbert function space $W^2(\Omega)$. These T_{Har}-like operators cannot be specified by conventional differential boundary conditions on the boundary of $\partial\Omega$, and may have non-empty essential spectra.

MSC: primary 35J40; 35J67; 35P05; secondary 32A36; 32A40; 47B25

Keywords: Linear partial differential equations; Self-adjoint partial differential equations; Spectral theory

1. Introduction

The *Harmonic operator* T_{Har} and the *Dirichlet operator* T_{Dir} are defined as self-adjoint linear partial differential operators (see the brief review in Section 2, with full details in the Memoir [3, Definitions 4.1 and 4.2]), which are extensions generated from any given linear elliptic partial differential expression of even order $n \geqslant 2$, satisfying certain reasonable conditions in a bounded region Ω with smooth boundary $\partial\Omega$, in Euclidean space $\mathbb{E}^r$ for $r \geqslant 2$.

Further investigations have shown that, see [6] for details, T_{Har} has a non-empty essential spectrum in the form of an eigenvalue of infinite multiplicity at the origin $0 \in \mathbb{C}$, in contradistinction to the known properties of T_{Dir} with its familiar discrete spectrum.

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0377-0427/$ - see front matter © 2006 Published by Elsevier B.V.
doi:10.1016/j.cam.2006.10.039

In this paper we embed both T_{Har} and T_{Dir} in an infinite family of self-adjoint extensions for an elliptic differential expression, in order to illuminate their inter-relationship within the total family of such self-adjoint extensions. These extensions are determined implicitly in the Stone-von Neumann Hilbert space theory, see [11, Chapter IV], or in the corresponding complex symplectic algebra theory of Everitt–Markus, see [3]. For simplicity of exposition we consider here only the special, but important, case of the elliptic expression for the classical Laplacian Δ, given by

$$\Delta f := \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}, \tag{1.1}$$

in the region of the unit disk Ω of the plane $\mathbb{E}^2$,

$$\Omega = \{(x, y) \in \mathbb{E}^2 : x^2 + y^2 < 1\} \tag{1.2}$$

in terms of the real rectangular co-ordinates (x, y). The boundary $\partial\Omega$ of Ω is then the unit circle in $\mathbb{E}^2$. However, many of the methods and the results apply also to the general case of elliptic partial differential expressions, as considered in [36].

All the linear operators considered here are defined on appropriate domains that are linear sub-manifolds of the Hilbert function space $L_2(\Omega)$, consisting of complex-valued square-integrable functions (or equivalence classes of such functions, in the usual manner) in the region Ω. Here, the notations for the scalar product $\langle\cdot,\cdot\rangle$ and norm $\|\cdot\|$ of $L_2(\Omega)$, and for other spaces, are given in Section 2, or more generally in [3, Appendix A, Parts I and II].

In particular, we consider the Sobolev Hilbert spaces in Ω, namely $W^l(\Omega)$ and $\overset{0}{W}{}^l(\Omega)$ for $l \in \mathbb{R}$, especially for $l \in [0, 2]$ (noting $W^0(\Omega) = L_2(\Omega)$); the definition of these spaces, see [36, Appendix A, Part I, Section 1], requires the introduction of weak or distributional partial derivatives. In addition we require the corresponding boundary Sobolev Hilbert spaces $W^l(\partial\Omega)$ in $\partial\Omega$, also for $l \in [0, 2]$. Likewise we consider the spaces of smooth functions, say $C^\infty(\Omega)$, $C^\infty(\bar{\Omega})$, $C_0^\infty(\Omega)$—and the corresponding spaces on the smooth compact manifold of the boundary $\partial\Omega$.

We require also certain trace operators which associate with any element $f \in W^2(\Omega)$ the values of $f|_{\partial\Omega}$ and the inward-drawn normal derivative $\partial f/\partial \mathbf{n}|_{\partial\Omega}$ on the boundary $\partial\Omega$. As an example, see [3, Section 2, (2.55), and Appendix A, (A.35) and (A.36)], the trace operator Tr_1 is a bounded linear surjection defined on $W^2(\Omega)$

$$\mathrm{Tr}_1 : W^2(\Omega) \to W^{3/2}(\partial\Omega) \times W^{1/2}(\partial\Omega) \quad \text{given by } f \to \left\{ f|_{\partial\Omega}, \left.\frac{\partial f}{\partial \mathbf{n}}\right|_{\partial\Omega} \right\}, \tag{1.3}$$

between the indicated Hilbert function spaces. The kernel of Tr_1 is given by

$$\mathrm{Ker}(\mathrm{Tr}_1) = \overset{0}{W}{}^2(\Omega) = \left\{ f \in W^2(\Omega) : f|_{\partial\Omega} = \left.\frac{\partial f}{\partial \mathbf{n}}\right|_{\partial\Omega} = 0 \text{ on } \partial\Omega \right\}, \tag{1.4}$$

so $\overset{0}{W}{}^2(\Omega)$ is a closed linear subspace of $W^2(\Omega)$.

2. Partial differential operators

The classical Laplacian Δ on the classical domain $C_0^\infty(\Omega) \subset L_2(\Omega)$, as in (1.1) and (1.2) above, defines a symmetric linear operator A, with a dense domain $D(A) := C_0^\infty(\Omega)$ in $L_2(\Omega)$, as follows:

$$Af := -\Delta f \quad \text{for all } f \in D(A), \tag{2.1}$$

noting the conventional negative sign. Moreover, the minimal closed symmetric extension T_0 of A, generated by Δ in $L_2(\Omega)$, is given by, see [3, Section 3, Theorem 3.2],

$$D(T_0) := \overset{0}{W}{}^2(\Omega) \subset L_2(\Omega) \quad \text{and} \quad T_0 f := -\Delta f \text{ for all } f \in D(T_0). \tag{2.2}$$

Here Δ defines a continuous map of $W^2(\Omega)$ into $L_2(\Omega)$ by means of weak or distributional partial derivatives.