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Conservation law with the flux function discontinuous in the space variable—II
Convex–concave type fluxes and generalized entropy solutions

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Abstract

We deal with a single conservation law with discontinuous convex–concave type fluxes which arise while considering sign changing flux coefficients. The main difficulty is that a weak solution may not exist as the Rankine–Hugoniot condition at the interface may not be satisfied for certain choice of the initial data. We develop the concept of generalized entropy solutions for such equations by replacing the Rankine–Hugoniot condition by a generalized Rankine–Hugoniot condition. The uniqueness of solutions is shown by proving that the generalized entropy solutions form a contractive semi-group in L^1. Existence follows by showing that a Godunov type finite difference scheme converges to the generalized entropy solution. The scheme is based on solutions of the associated Riemann problem and is neither consistent nor conservative. The analysis developed here enables to treat the cases of fluxes having at most one extrema in the domain of definition completely. Numerical results reporting the performance of the scheme are presented.

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Keywords: Conservation law; Generalized Rankine–Hugoniot condition; Convex-concave type fluxes; Generalized entropy solutions; Godunov type finite difference scheme

1. Introduction

We are interested in the following single conservation law in one space dimension:

$$u_t + (f(k(x), u))_x = 0,$$
$$u(x, 0) = u_0(x), \tag{1}$$

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where the flux f depends on the space variable through a coefficient k which may be discontinuous. The dependence can be of the multiplicative type given by

$$u_t + ((k(x)f(u))_x = 0,$$
$$u(x,0) = u_0(x) \quad (2)$$

but we are interested in the simplest case—the so-called "two flux" case given by

$$u_t + (H(x)f(u) + (1 - H(x))g(u))_x = 0,$$
$$u(x,0) = u_0(x), \quad (3)$$

where f and g are Lipschitz continuous functions and H is the Heaviside function. We remark that the analysis of (3) will be the building block in the analysis of (1). For the rest of this paper, we shall be concerned with (3).

The conservation law (1) occurs in several models in Physics and Engineering. In particular, it arises in two phase flow in a heterogeneous porous medium used in petroleum reservoir simulation. The unknown u generally denotes the saturation of one of the phases and the flux is given in terms of the Darcy velocities. The change of rock type leads to a variation of the absolute permeability of the medium and relative permeabilities of the phases and results in discontinuities in the flux function. For further details, refer to [14,15].

Another physical model where equations of the form (1) arise are in the modeling of an ideal clarifier thickener unit used in the waste water treatment plants and in the paper industry. Discontinuities in the flux arise from the modeling of the feed inlet leading to a separation of the mixture into upward and downward flows. For a detailed description refer to [8,6]. Eq. (1) also arises in modeling traffic flow on highways with changing surface conditions (see [26]) and in ion etching used in the Semiconductor industry (see [27]). For detailed account of various applications of (1), see [24].

As is standard for conservation laws, we have to look for a suitable form of weak solutions and augment them with extra admissibility criteria or entropy conditions for uniqueness and stability. The development of a proper entropy framework for equations of the type (1) is a major challenge. The equations of the type (1) have been studied extensively over the last decade from both the analytical and the numerical points of view.

In [10,11], Gimse and Risebro used a "minimal variation" condition at the interface ($x = 0$) and showed uniqueness of solution for the Riemann problem associated with (3). In [8,9], Diehl imposed a different condition (Γ condition) at the interface to select solutions. Some results regarding uniqueness of solutions for (2) were obtained by Klingenberg and Risebro in [22]. Karlsen, Risebro and Towers have proposed an entropy formalism for (1) (including a degenerate parabolic term) in [18]. They used a modified Kruzkhov type entropy condition and showed that the entropy solutions formed an L^1 stable semi-group under the assumption that the traces of the solution exists at the interface and the fluxes satisfy a certain "crossing condition". Their analysis was extended to the case of time dependent coefficients in [7].

Concurrently, several existence results for the entropy solutions have been obtained in a series of papers. They use regularization of coefficients as in [16], some are based on front tracking as in [11,22,21,7] while others used numerical schemes of the Godunov or Enquist-Osher type as in [29,30,17,6] and of the Lax–Friedrichs type as in [19].

Independently, Adimurthi and Gowda studied (3) by considering the corresponding Hamilton Jacobi equation in [1]. Under the assumptions that the fluxes are strictly convex and have super linear growth, they were able to obtain an explicit Hopf–Lax type formula for the solutions. Using this formula, they were able to obtain explicit solutions of the Riemann problem associated with (3) and derive a different entropy condition at the interface which essentially amounted to the exclusion of undercompressive waves at the interface. They were able to show that the entropy solutions formed an L^1 contractive semi-group.

In [3], Adimurthi, Jaffre and Gowda relaxed the hypothesis on the fluxes by requiring that the fluxes can have at most one minima (one maxima) in the domain. They developed a Godunov type finite difference (finite volume) scheme and showed that the approximations converged to the entropy solution of (3). In [25], the author was able to handle the case when f has one maxima (resp. minima) and g has one minima (resp. maxima) in the domain (under the additional condition that the fluxes f and g intersect at the endpoints of the domain), obtained uniqueness without any additional entropy conditions at the interface and obtained existence results with a Godunov type scheme.

In a recent work [4], the authors have proposed a new entropy framework for equations of the type (3)—the so-called *optimal entropy solutions*. Under the assumptions that f and g are both of the convex (concave) type (for definitions, check from later in this section) or f is of convex type and g is of the concave type in the domain of definition,