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Multigrid Newton–Krylov method for radiation in diffusive semitransparent media

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Abstract

We present a fast multigrid solver for simplified P_N (SP_N) approximations to the diffusive radiation in non-grey semitransparent media. The method consists on reformulating the equations as a nonlinear fixed point problem in the temperature only. Given a mesh hierarchy, time and space discretizations are performed using second-order implicit and finite differencing methods, respectively. At each mesh level, a Newton–Krylov algorithm is applied to the discrete equations. As a smoother on the coarse meshes we propose the Atkinson–Brakhage operator. Numerical results are shown for glass cooling process using different geometry enclosures. The SP_N approximations capture the correct asymptotic behavior of the numerical solution with a computational cost lower than using the full radiative transfer equations.

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1. Introduction

Let Ω be a geometrical domain in $\mathbb{R}^d$ ($d = 1, 2$ or 3) with smooth boundary $\partial\Omega$ of an absorbing and emitting semitransparent material with a given initial temperature distribution

$$T(0, \mathbf{x}) = T_0(\mathbf{x}). \tag{1}$$

The heat conduction in the medium Ω is described by the energy equation

$$\rho c \frac{\partial T}{\partial t} - \nabla \cdot (K \nabla T) = -\int_{\nu_1}^{\infty} \int_{\omega=4\pi} \kappa(\nu)(B(T, \nu, n_{\mathrm{m}}) - I)\, \mathrm{d}\omega\, \mathrm{d}\nu, \tag{2}$$

where ρ is the density, c denotes the specific heat capacity, $\mathbf{x}$ the position vector, t the time, T the temperature, K the thermal conductivity and κ the absorption coefficient. On the boundary the heat flux $K\mathbf{n}(\hat{\mathbf{x}}) \cdot \nabla T$ is defined by heat

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convection and diffuse surface radiation

$$K\mathbf{n}(\hat{\mathbf{x}})\cdot\nabla T+\lambda(T-T_{\mathrm{b}})=\alpha\pi\int_0^{v_1}(B(T_{\mathrm{b}},v,n_{\mathrm{b}})-B(T,v,n_{\mathrm{b}}))\,\mathrm{d}v, \tag{3}$$

where λ is the convective heat transfer coefficient, T_{b} is a given temperature of the surrounding, $\mathbf{n}(\hat{\mathbf{x}})$ denotes the outward normal in $\hat{\mathbf{x}}$ with respect to $\partial\Omega$ and α the mean hemispheric surface emissivity in the opaque spectral region $[0, v_1]$, where radiation is completely absorbed. In (3), n_{b} and n_{m} are refractive indices of surrounding medium and semitransparent material, respectively. $B(T, v, n)$ is the spectral intensity of the black-body radiation given by the Planck's function in a medium with refractive index n

$$B(T,v,n)=\frac{2h_{\mathrm{P}}v^3}{c_0^2}n^2(\mathrm{e}^{h_{\mathrm{P}}v/k_{\mathrm{B}}T}-1)^{-1}. \tag{4}$$

Here h_{P}, k_{B} and c_0 are Planck's constant, Boltzmann's constant and the speed of radiation propagation in vacuum, respectively [10].

The spectral intensity $I(\mathbf{x}, \omega, v)$ at the space point $\mathbf{x}$, within the frequency v and along the direction ω, is obtained from the radiative transfer equation

$$\forall v>v_1:\quad \omega\cdot\nabla I+\kappa(v)I=\kappa(v)B(T,v,n_{\mathrm{m}}). \tag{5}$$

At the boundary we consider transmitting and specular reflecting condition

$$I(\hat{\mathbf{x}},\omega,v)-\varrho(\mathbf{n}\cdot\omega)I(\hat{\mathbf{x}},\omega',v)=(1-\varrho(\mathbf{n}\cdot\omega))B(T_{\mathrm{b}},v,n_{\mathrm{b}}),\qquad \mathbf{n}(\hat{\mathbf{x}})\cdot\omega<0, \tag{6}$$

where $\omega'=\omega-2(\mathbf{n}\cdot\omega)\mathbf{n}$ is the specular reflection of ω on $\partial\Omega$, and $\varrho\in[0,1]$ is the reflectivity obtained according to the Fresnel and Snell laws [16]. Thus, for an incident angle θ_{m} given by $\cos\theta_{\mathrm{m}}=|\mathbf{n}\cdot\omega|$ and Snell's law

$$n_{\mathrm{b}}\sin\theta_{\mathrm{b}}=n_{\mathrm{m}}\sin\theta_{\mathrm{m}},$$

the reflectivity $\varrho(\mu)$, $\mu=|\mathbf{n}\cdot\omega|$, is defined as follows:

$$\varrho(\mu)=\begin{cases}\dfrac{1}{2}\left(\dfrac{\tan^2(\theta_{\mathrm{m}}-\theta_{\mathrm{b}})}{\tan^2(\theta_{\mathrm{m}}+\theta_{\mathrm{b}})}+\dfrac{\sin^2(\theta_{\mathrm{m}}-\theta_{\mathrm{b}})}{\sin^2(\theta_{\mathrm{m}}+\theta_{\mathrm{b}})}\right) & \text{if } |\sin\theta_{\mathrm{m}}|\leqslant\dfrac{n_{\mathrm{b}}}{n_{\mathrm{m}}},\\ 1 & \text{otherwise.}\end{cases} \tag{7}$$

We assume that $n_{\mathrm{m}}>n_{\mathrm{b}}$ and the hemispheric emissivity α is related to the reflectivity ϱ by

$$\alpha=2n_{\mathrm{m}}\int_0^1(1-\varrho(\mu))\,\mathrm{d}\mu.$$

There is a vast literature dealing with numerical methods for the radiative heat transfer (RHT) equations (1)–(6), see [16] for a survey. These equations have been the key to understand the temperature distribution on many semitransparent materials. As an example, the above equations have been widely used to predict the temperature distribution during the cooling process of glass which has direct effect on the quality of the product. Moreover, numerical experiments on semitransparent materials have shown that heat transfer cannot be estimated only by conduction but also by radiation. For instance, in the annealing process, glass temperature is higher than 700 K and at this temperature radiative transfer dominates conduction.

The main difficulties raised when solving numerically the RHT equations lie essentially on the large set of dependent unknowns, the coupling between the radiative transfer and the heat conduction, and the specularly reflecting boundary conditions. The most accurate procedures available for computing RHT in semitransparent materials are the zonal and Monte Carlo methods [11]. However, these methods are not widely applied in comprehensive radiative transfer calculations due to their large computational time and storage requirements. Also, the equations of the radiation transfer are in non-differential form, a significant inconvenience when solved in conjunction with the differential equations of flow and conduction. For this reason, numerous investigations are currently being carried out worldwide to assess computationally efficient methods. The present work deals with the design of such methods.

In this paper, we consider the $\mathrm{SP_N}$ approximations to the RHT problem. The $\mathrm{SP_N}$ approximations were first proposed in [5] and theoretically studied in [9]. In [8,15] the $\mathrm{SP_N}$ approximations have been extensively studied for radiative